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On the Poisson Follower Model

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On the Poisson Follower Model

by

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Dedicated to my family.

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Last few years have been made easy with my furry friend Ricci (not to be confused with the mathematician Gregorio Ricci-Curbastro, inventor of tensor calculus), who is always my number 1 fan. But above all, I'm grateful to my family for always being there for me and encouraging my interests. This thesis is dedicated to them.

On the Poisson Follower Model

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This dissertation presents studies of dynamics over the Poisson point process. In particular, we study a special case of Hegselmann-Krause Dynamics [1] over $\mathbb{R}^2$. Chapter 1 is a brief introduction to the thesis and its structure.

Chapter 2 introduces the notation, the definitions and examples of phenomena of interest.

In Chapter 3, we go deeper in analyzing the phenomena described by calculating frequency of these phenomena. A system of quadratic inequalities will be introduced to allow one to calculate the probabilities of the events pertaining to this dynamics using methods from integral geometry.

Chapter 4 uses percolation arguments to prove the absence of percolation at step 1.

In Chapter 5, we provide geometric results of independent interest pertaining to the Follower Dynamics.

In Chapter 6, we discuss the limiting behavior of this process and include some more simulations.

In Chapter 7 we propose future steps and discuss more general Hegselmann-Krause Dynamics.

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Chapter 1

Introduction

> "Where shall I begin, please your Majesty?" he asked. "Begin at the beginning," the King said gravely.
>
> Lewis Carroll, Alice's Adventures in Wonderland and Through the Looking-Glass

We propose a mathematical model that looks into opinion evolution and long term opinion dynamics of a countable collection of agents. Each agent has an $\mathbb{R}^2$-valued opinion and can only be influenced by the agent whose opinion is closest to its opinion. The agent it follows is its *leader*. The follower/leader connections form a directed graph, where the direction goes from the follower towards the leader. At every time step of the dynamics, each agent's opinion moves halfway to that of its leader. Thus, our model is a variant of the Hegselmann-Krause dynamics [1] extended to a particle system setting [4].

The Hegselmann-Krause model has been studied extensively in the literature [18, 19, 20, 21], with still many interesting problems to solve. This model is just one among the large class of opinion dynamics models. Other examples of proposed models can be found in [22, 23, 24]. A survey on opinion dynamics can be found in [25].

Initially, we intended to study the full Hegselman-Krause model using

Palm calculus. This was not studied in the literature to the best of our knowledge. However, the initial model is more complex than initially thought. We hence simplified the problem by only keeping the follower aspect. This new model still captures the essence of some social network interactions, like e.g. Instagram and Twitter, where the asymmetric leader/follower dynamics is central.

We think of a group of agents that are part of the same connected component of this graph as a *party*. As a result of this dynamics, at each step, agents either stay in their party or switch positions within a party, or change party. Equivalently, under a step of the dynamics, a party itself can either stay the same, split, loose, or gain new agents. We are interested in the fate of the agents opinions and that of the parties as time evolves and also as it tends to infinity.

If the initial opinions form a random point process, the directed graph is generated over a random point set, and hence it is a random graph. This random graph then evolves over time based on the prescribed dynamics. Due to the fact that each agent may follow different leaders at different times, the graph structure may change over time. In this work, we focus on two things: proving local geometric properties using stochastic geometry and deriving global asymptotic results. We look at the opinion of an agent as a point in $\mathbb{R}^2$ and Euclidean distances between points represent how close their opinions are. Points of the initial configuration are sampled according to a Poisson point process over $\mathbb{R}^2$. We draw a directed edge from each point to its closest neighbor, its leader. The connected components of this directed graph form what are called *Poisson descending trees.* An example of such a tree is shown in Figure 1.1. It is well known that these Poisson descending trees are a.s.

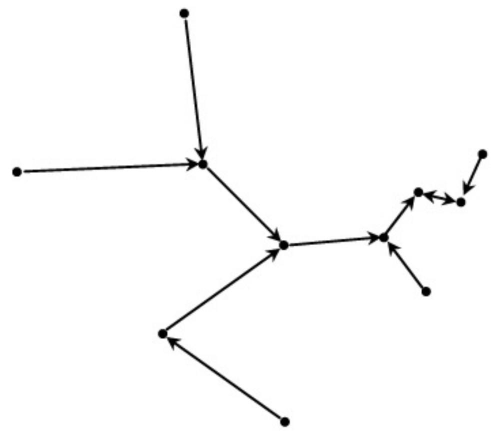


Figure 1.1: Example of a Poisson descending tree

finite [30]. Then we apply the dynamics described above, namely, every point moves halfway towards its leader.

Here are examples of questions arising in this context. After many iterations, what happens in this system? Can there be an infinite chain of followers forming at any step? Does the dynamics converge? To what limit?

Before we begin the analysis, we list several phenomena observed in this dynamics that are discussed throughout the thesis:

- One point can be a leader to several points, a number that depends on the dimension. In dimension two, for all point configurations, a point can be a leader for up to 6 followers. This number is at most 5 a.s. in the Poisson case. This is related to the kissing number [2]. Recall that the kissing number is defined as the number of non-overlapping unit balls that can be arranged in such a way that they each touch another given unit ball. Each initial descending tree has exactly one "loop", i.e., only one pair of mutually closest neighbors who will be referred to as *the* *ultimate leader pair of the tree.*

- The Poisson descending tree is also known as the Poisson Nearest Neighbor graph and has been studied extensively [14]. In particular, we know the initial fraction of ultimate leader pairs in such a system [14].

- If two points form an ultimate leader pair, then, after one step, they merge to be in the same opinion and will never change opinion/position.

- At each step of the Follower Dynamics, an agent either keeps the same leader or follows another agent. Similarly, a leader at any step can keep its followers, gain, or lose some followers. One interesting phenomenon we will discuss in the subsequent sections is that two agents in a follower/leader pair can switch their roles in one step of the dynamics.

- Some configurations of points lead to a *party break*, i.e., a situation where a point leaves one party and becomes a member of another one.

- As time progresses, there is an increasing fraction of points that belong to an ultimate leader pair. Points that aren't ultimate leaders of any order will be called *ultimate followers*.

The Poisson Follower Point Process of order n is the point process obtained at the n^{th} iteration of this dynamics when starting from a Poisson Point Process. At the n^{th} iteration, the agents or points are the vertices of a graph called the *Poisson Follower graph of order n*.

Here are the main aims for this work. We want to show that, for all $n \geq 0$, the Poisson Follower graph of order n has parties which are all of finite size a.s. We want to know the shape and moments of the limiting objects (point processes and graphs). We want to estimate the frequency of some of the phenomena described above. Furthermore, we want to determine the distribution of the ultimate leader pairs at any given step and explain mathematically the distribution we observe in simulations. We also analyze parties as trees and we want to study the convergence of these trees.

Let us stress that this general aims are only partially reached in the thesis. Future research is included in Chapter 7.

The contributions can be divided in three broad categories, each of which will be discussed extensively in the thesis.

1. Characterization of the phenomena pertaining to the Follower Dynamics starting from Poisson point processes and proof of the fact that the spatial frequency of each such phenomenon can be evaluated in terms of integrals over a semi-algebraic sets of the Euclidean plane.

2. Proof of the absence of percolation for different steps of this dynamics. This result leverages the theory of dependent percolation over the square $\mathbb{Z}^2$ lattice and the mass transport principle.

3. Study of the limiting behavior of this dynamics when the number of steps tends to infinity. We will in particular identify two sub point processes, that of points converging to a limiting point in total variation, and that of points converging weakly but not in variation. We will also identify the structure of the limiting graphs of this dynamics.

In Chapter 2, we give the notation, the definitions and examples of the phenomena observed in this dynamics. In Chapter 3, we go deeper in analyzing the phenomena described by calculating their relative frequencies. A system of quadratic inequalities will be introduced to allow one to calculate the probabilities of the events pertaining to this dynamics via integral geometry. Chapter 4 uses percolation arguments to prove the absence of percolation at step 0 and step 1. In Chapter 5, focuses on the local geometric properties of the Poisson Follower model. Then, in Chapter 6, we discuss the limiting behavior of this process and include some simulations. In Chapter 7 we summarize the results and list future steps. Chapter 7 also looks at more complex problems, in particular general Hegselmann-Krause dynamics and discusses difficulties with getting general results under such dynamics. Finally the Appendix includes omitted details from Chapter 3 and proofs of previously known theorems relevant to our problem.

Chapter 2

Notation and Definitions

> "Must a name mean something?" Alice asked doubtfully.
> "Of course it must," Humpty Dumpty said with a short laugh; "my name means the shape I am - and a good handsome shape it is, too. With a name like yours, you might be any shape, almost."
>
> Lewis Carroll, Alice in Wonderland

In this Chapter, we introduce the notation, background material, and terminology that is used throughout the thesis. Section 2.1 contains the notation and some assumptions. General references on point processes studied here are included in [9, 12, 5]. Section 2.2 gathers illustrations and explanations of different phenomena observed in this dynamics. Its main goal is to give an intuition of how the Poisson Follower dynamics behaves and what are the different phenomena one can expect.

2.1 Notation and Terminology

Framework: Let $|| \bullet ||$ denote the Euclidean norm on $\mathbb{R}^2$ and $B(A, r)$ the open ball of radius r and center A.

Let Λ be a a locally compact Polish space equipped with its Borel σ-algebra $\mathcal{B}$ and Radon measure ν. A point process Φ on Λ is a random locally finite subset of Λ. One can also view Φ as a random counting measure on Λ, having the form $\Phi = \sum_{i\in\mathbb{N}} \delta_{S_i}$, where $\{S_i\}_{i\in\mathbb{N}}$ is a countable collection of points in Λ with no accumulation points.

Now we give some basic properties related to point processes. A point process is called *simple* if $\Phi(\{x\}) \leq 1$ for all $x \in \Lambda$. A point process whose distribution is invariant under translations is called *stationary*. The *intensity measure* of a point process Φ is the measure on Λ defined by

$$\lambda(B) = \mathbb{E}[\Phi(B)],\ B \in \mathcal{B}(\Lambda).$$

Definition 2.1. *For any counting measure $\mu = \sum_{i\in\mathbb{N}} \delta_{x_i}$ and $n \in \mathbb{N}$, its n-th power in the sense of products of measures is*

$$\mu^n = \sum_{(i_1,\ldots,i_n)\in\mathbb{Z}^n} \delta_{(x_{i_1},\ldots,x_{i_n})}.$$

Define the n-th factorial moment measure as the following counting measure on $(\mathbb{R}^2)^n$.

$$\mu^{(n)} = \sum_{(i_1\neq\ldots\neq i_n)} \delta_{(x_{i_1},\ldots,x_{i_n})}.$$

Additionally, moment measures are an important object because they give information on the distribution of a point process.

Definition 2.2. *Moment measures. For a point process Φ on a l.c.s.h. space $\mathbb{R}^2$, let Φ^n be the n-th power of Φ and $\Phi^{(n)}$ be the n-th factorial moment of Φ. We call M_{Φ^n} the n-th moment measure (the first moment measure is the mean measure) of Φ and $M_{\Phi^{(n)}}$ the n-th factorial moment measure.*

Moment measures give important average structural characteristic of the process, such as level of clustering or repulsion.

The point process that is studied the most is the Poisson point process, and our dynamics starts from it as well. For the Poisson point process, all points are stochastically independent, and the number of points in a bounded set follows a Poisson distribution. The formal definition is as follows.

Definition 2.3. *(Poisson point process) Let Λ be a locally finite measure on l.c.s.h. space $\mathbb{G}$. A point process Φ is said to be Poisson with intensity measure Λ if for all pairwise disjoint sets $B_1, ..., B_j \in \mathcal{B}(\mathbb{G})$, the random variables $\Phi(B_1), ..., \Phi(B_j)$ are independent Poisson random variables with respective means $\Lambda(B_1), ..., \Lambda(B_j)$; i.e. $\forall m_1, ..., m_j \in \mathbb{N}$,*

$$\mathbb{P}(\Phi(B_1) = m_1, ..., \Phi(B_j) = m_j) = \prod_{i=1}^{j} \frac{\Lambda(B_i)^{m_i}}{m_i!} e^{-\Lambda(B_i)}. \tag{2.1}$$

Definition 2.4. *(Homogeneous Poisson point process on $\mathbb{R}^2$) If Φ is a Poisson point process on $\mathbb{R}^2$ with intensity measure $\Lambda(dx) = \lambda \times dx$ where $\lambda \in \mathbb{R}_+^*$ and dx denotes the Lebesgue measure, then Φ is called a homogeneous Poisson point process of intensity λ.*

In addition, from the definition it follows that the restriction of a Poisson point process Φ of intensity measure Λ to some $B \in \mathcal{B}(\mathbb{G})$, i.e. the point process $\Phi|_B(\cdot) = \Phi(\cdot \cap B)$, is a Poisson point process of intensity measure $\Lambda|_B = \Lambda(\cdot \cap B)$.

Throughout the thesis, we denote as $\Phi = \sum_{i \in \mathbb{Z}} \delta_{x_i}$ a homogeneous Poisson point process on $\mathbb{R}^2$ with intensity λ, which serves as initial condition to the dynamics. Note that $\Phi = \Phi_0$. For all $n \geq 0$, the Poisson Follower point process of order n will be denoted by Φ_n.

For two points $x, y \in \Phi$, we denote by $B(x \to y)$ the open ball with center x and radius $d(x, y)$, which is the distance between x and y.

Here is now some vocabulary and notations making it easier to discuss our dynamics. Consider a counting measure on $\mathbb{R}^2$ or $\mathbb{R}^d$. Take two points A and B in this counting measure. If the closest point to A is B, we say that A *follows* B. Other equivalent statements are, "A is a *follower* of B", or "B is a *leader* of A", and "B is *followed* by A", etc. We will also use notation $A \to B$ to mean that A follows B. If B follows C, and A follows B, we call C a *leader of order two* of A. The leader of A will be denoted by $\mathscr{A}(A) = B$. The leader of order k of A will be denoted by $\mathscr{A}^k(A)$. Two agents A and B are part of the same *follower chain* if there is a k such that $\mathscr{A}^k(A) = B$, or l such that $\mathscr{A}^l(B) = A$. Note that this is an equivalence relation. We call *ultimate leader pair* agents that follow each other.

Because of scale invariance of the Poisson point process, we will assume without loss of generality that $\lambda = 1$ in subsequent chapters.

Palm Theory Informally, the Palm measures of a point process Φ at a point $x \in \Lambda$ is the probability measure of Φ conditioned on having a point at location x. We shall define the Palm measure now. For a more detailed discussion on the matter, see [40, 9].

Let $(\Omega, \mathcal{A}, \{\theta\}_{t\in\mathbb{R}^2}, \mathbb{P})$ be a stationary framework and Φ a point process compatible with the flow $\{\theta\}_{t\in\mathbb{R}^2}$ implying Φ is stationary. Let λ be the intensity of Φ. The Palm measure $\mathbb{P}^0$ associated with Φ is defined on $(\Omega, \mathcal{A})$ by

$$\mathbb{P}^0(A) := \frac{1}{\lambda}\mathbb{E}\Big[\int_B 1_A \circ \theta_t \Phi(dt)\Big],$$

for any bounder Borel set B with volume one.

We will use the following more general theorem in later chapters.

Theorem 2.1. *Campbell-Little-Mecke-Matthes (C-L-M-M) Let* $(\Omega, \mathcal{A}, \{\theta\}_{t\in\mathbb{R}^2}, \mathbb{P})$ *be a stationary framework and* Φ *be a random measure compatible with the flow* $\{\theta\}_{t\in\mathbb{R}^2}$*. Then for any measurable function* $f : \mathbb{R}^d \times \Omega \to \overline{\mathbb{R}}_+$*,*

$$\mathbb{E}\Big[\int_{\mathbb{R}^2} f(x, \theta_x\omega)\Phi(dx)\Big] = \lambda \int_{\mathbb{R}^2} \mathbb{E}^0[f(x,\omega)]dx.$$

Mass Transport formula

Theorem 2.2. *Theorem 6.1.34. in [9]. Let* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbb{P})$*, be a stationary framework and* Φ, Φ' *be random measures on* $\mathbb{R}^d$ *compatible with the flow* $\{\theta_t\}_{t\in\mathbb{R}^d}$ *and having respective intensities* $\lambda, \lambda' \in \mathbb{R}*_+$*. Then, for all measurable functions* $g : \mathbb{R}^d \times \Omega \to \bar{\mathbb{R}}_+$*,*

$$\lambda\mathbb{E}^0\Big[\int_{\mathbb{R}^d} g(y,\omega)\Phi'(dy)\Big] = \lambda'\mathbb{E}^{0'}\Big[\int_{\mathbb{R}^d} g(-x, \theta_x\omega)\Phi(dx)\Big],$$

where $\mathbb{E}^0$ *and* $\mathbb{E}^{0'}$ *are the expectations with respect to the Palm probabilities of* Φ *and* Φ'*, respectively. The above formula is called the mass transport formula.*

In the thesis we use the following interpretation of the Mass Transport formula (Theorem 6.1.34. in [9]): Let $m(x, y, \omega)$, be a measurable function on $\mathbb{R}^d \times \mathbb{R}^d \times \Omega$ interpreted as the amount of mass sent from x to y on the event ω. We assume that m is compatible with the flow in the following sense

$$m(x, y, \omega) = m(x - t, y - t, \omega),\ x, y, t \in \mathbb{R}^d.$$

Define $g(y, \omega) := m(o, y, \omega)$ as the amount of mass sent from the origin 0 to y on the event ω. Then by the compatibility of m we have the mass transport formula written as

$$\lambda\mathbb{E}^0\Big[\int_{\mathbb{R}^d} m(0, y, \omega)\Phi'(dy)\Big] = \lambda'\mathbb{E}^{0'}\Big[\int_{\mathbb{R}^d} m(x, 0, \omega)\Phi(dx)\Big],$$

which is interpreted as by saying that the proportion between the expected total mass sent from the typical point of Φ (located at the origin under $\mathbb{E}^0$) to all points of Φ' and the expected total mass received by the typical point of Φ' (located at the origin under $\mathbb{E}^{0'}$) from all points of Φ is equal to the proportion of the point processes intensities λ' to λ.

2.2 Definitions and Examples

In this section we give some basic definitions to be used throughout the document. We also define further the phenomena of interest and give some illustrations.

Forward and Backward Sets We denote by $For(x, \Phi)$ the leader set of all orders of x in Φ, and we will call it the *forward set of x*. $Back(x, \Phi)$ denotes the follower set of all orders of x in Φ, and we call it the *backward set of x*. See Figure 2.1.

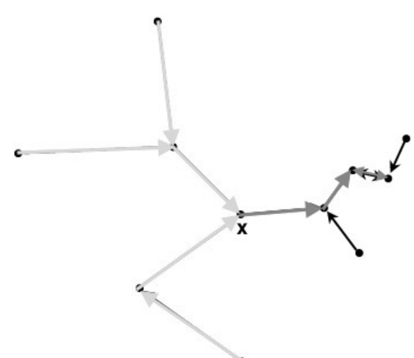


Figure 2.1: Representation of the forward and backward sets of an agent. The forward set of x is connected to x with dark gray arrows and the backward set of x is with light gray ones. The agents connected by black arrows are in the follower party of x but not in the backward or forward set of x.

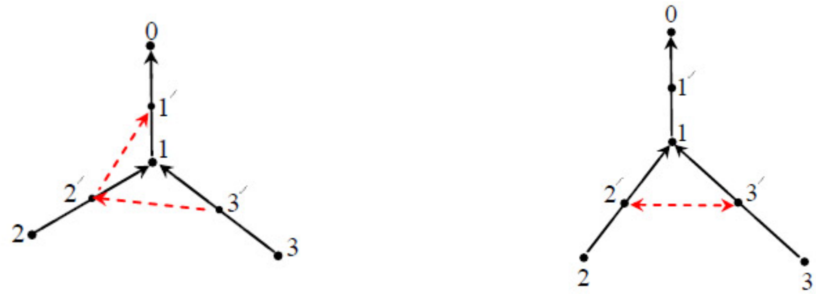


Figure 2.2: **Left:** Example of a Follower Loss/Follower Gain. The initial positions are shown in black. The step 1 is in red, with the new positions denoted with the prime '. **Right:** Example of a ultimate leader pair of order 1. Same convention as in the left image

Follower party We call *follower party* any set of points belonging to one connected component of the follower graph. Namely, two points x and y belong to the same follower party if there exists a point z (not necessarily different from x or y), such that both x and y belong to the backward set of z. $x, y \in Back(z, \Phi)$.

One example can be found in Figure 2.1. Since the initial agent distribution is Poisson at step 0, every follower party is finite almost surely at step 0 [30].

Looking from any agent (point) of the system, here is a classification of situations that can happen to this agent when applying the dynamics.

"Out" edges The "out" edge of this point, is either kept (*leader keep*) or changed (*leader swap*). Notice that these two situations are mutually exclusive since each point gets only one leader at any given time. For example in Figure 2.2, Agent 3 swapped its leader from 1 to 2, as a result of the dynamics.

"In" edges For the "in" edges of the point, one can have a *follower loss*, a *follower gain* or a *follower keep*. Notice that in this case, all the situations can happen at the same time, since a point can have several followers. In the left Figure 2.2, Agent 1 lost its follower, Agent 3. Agent 2 gained a follower, namely Agent 3. Agent 1 kept Follower 2.

Ultimate leader pairs A special case is when a new pair of ultimate leaders is formed. An example of this is shown in the right Figure 2.2. Let us call initial ultimate leader pairs, ultimate leader pair of order 0. New ultimate leader pairs, forming at step 1 will be called ultimate leader pairs of order 1, etc.

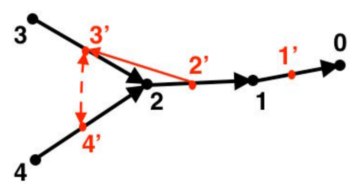


Figure 2.3: Example of a follower inversion.

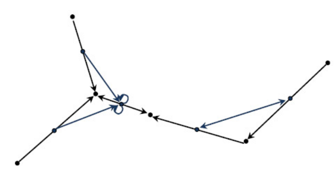


Figure 2.4: Example of party fission

Follower inversion Another special case is that when, after one step, the leader of an agent becomes its follower. We call such a situation a *follower inversion*. We do not count situations when the leader and follower become an ultimate leader pair at step 1 in such a follower inversion. An example of follower inversion is shown in Figure 2.3. The initial positions are shown in black. The positions at the step 1 are shown in red, with the new positions denoted with the prime, like for example $1'$. Agent 3 was following agent 2 initially, but at step 1, Agent 2 follows Agent 3, and Agents 3 and 4 form an ultimate leader pair.

Looking at parties, several situations can also happen.

Party fission The situation when there is a new ultimate leader pair created within a party will be called a *party fission*. Figure 2.4 gives an example of a party fission. When a party increases its cardinality due to a swap of an agent from another party, we speak of *party gain*. The opposite event, i.e., a decrease in cardinality due to a swap of an agent from a party toward another preexisting party is called a *party loss*. A party can have both losses and gains at a given step.

Party restructuring By *party restructuring*, we mean a situation where there is a swap in the party which does neither lead to a fission nor to a loss in the party, like the situation illustrated in Figure 2.2 right. A *stable party* is a party in which there are no swaps (and no losses or gains) at any future time step. Stable parties are discussed in Section 6.2.

Party swap The situation where an agent changes its follower party will be called *a party swap*. An example of a party swap is shown in Figure 2.5. Another example of a party swap is shown in Figure 2.6, where agent z and its followers have a party swap.

4 body swap In order to analyze the frequency of party swaps, it helps to focus on the necessary conditions for one agent to leave its party. There has to be an agent that is swapping to a leader from another party in order

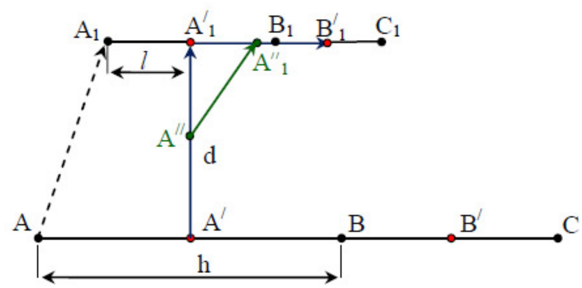


Figure 2.5: Example of a 4 body swap. A',B', A_1' and B_1' are the positions of agents in the next time step. Positions in the second time step are denoted by A" and A_1" and connections are shown in green

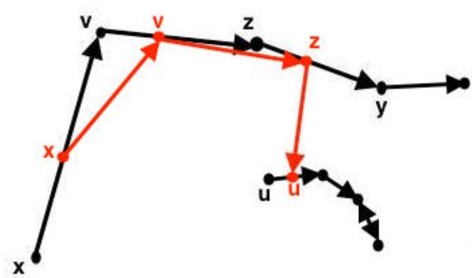


Figure 2.6: Example of a party swap. Initial positions are shown in black and step 1 is in red. Agent z swaps leaders and parties. Together with z, agents x and v also have a party swap.

for the party swap to occur. See Figure 2.6 as an example of a party fusion. In particular, for an agent z to swap leader, we need to be able to compare the distances between the agent z and its leader y, and the distance between the agent z and the potential leader at step 1, u. Thus we need to know the positions of the agent and the two swapping leaders at step 1, i.e., whether $d(z, u, \Phi_1) < d(z, y, \Phi_1)$? This is why we define a *4 body swap* to be a leader swap that involves 4 different agents: A, B, A_1, and B_1, with, A following B, and A_1 following B_1 at step 0. Then, at step 1, A follows A_1 or A_1 follows

A, i.e., a leader swap occurs. However, in order to know whether A follows A_1 at step 1, we need to know where B and B_1 moved to. Therefore, for the analysis, we also need the positions of the leaders of B and B_1, C and C_1, respectively. See Figure 2.5 for an illustration. Using the notation of Figure 2.5, we want $d(A', A_1') < d(A', B')$. A 4 body swap does not imply a party swap necessarily. In Section 3.2 we calculate the frequency of this event.

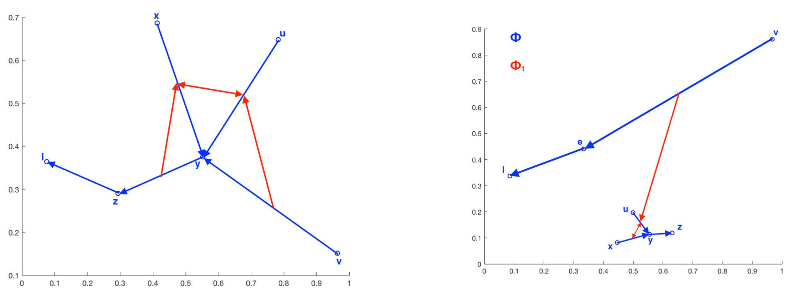


Figure 2.7: **Left:** Initial positions and connections are shown in blue. Step 1 is shown in red. Agent y starts following x after 1 step, i.e., we see an example of a follower inversion. Agents x and u become ultimate leader pair of order 1. **Right:** Example of a party swap, where v swaps parties and agents u and x form an ultimate leader pair of order 1.

Phenomena are not excluding of each other. Under the follower dynamics, we observe all the phenomena described in this section and it can happen that after one step, several events happen at the same time on a given set of agents. For example, in the left Figure 2.7, we observe both a follower inversion and an ultimate leader pair of order 1 being created. The left of Figure 2.7 also gives an example of party restructuring and party fission since there are 2 pairs of ultimate leaders. In the right Figure 2.7 we observe both a party swap and the formation of an ultimate leader pair of order 1. The right

Figure 2.7 is also an example of a simultaneous party restructuring and party swap.

Chapter 3

Estimates of Phenomena Frequencies

"Why," said the Dodo, "the best way to explain it is to do it."

Lewis Carroll, Alice in Wonderland

In this chapter, we classify the phenomena we observe and devise a systematic way to calculate the spatial frequencies of the corresponding events using integral geometry. We show that this leads to the evaluation of integrals of certain exponential functions over certain semi-algebraic sets. Semi-algebraic sets are the subsets of $\mathbb{R}^n$ which are finite unions of real solution to polynomial systems of equations and inequalities with coefficients in $\mathbb{R}$ [49]. This approach is applicable to each phenomenon listed above.

First, as a warm up, in Section 3.1, we look at all pairs of points and show how to calculate how frequently they are each others closest neighbor, i.e., we calculate the density of ultimate leader pairs of order zero. This is a known result [14], but derived here in a novel way in order to present the method. Then, we extend this approach to larger sets of points in order to look at densities of party swaps of higher order. Then, in Section 3.2, we also show how to calculate the frequency of party inversions and 4 body swaps. These estimates are obtained in two ways, one is through numerical analysis of integrals, and the other using the Ergodic theorem on simulations. In order to verify our findings, we compare our results to the simulation of this process. In Section 3.3, we explain both methods in depth and discuss the numerical values obtained.

Finally, we explain how to generalize this in a natural way in Section 3.4.

3.1 Integral Geometry Estimates of Densities of Ultimate Leader Pairs

In this subsection, we give an integral geometry representation of ultimate leader pairs and then give numerical estimates. First, we calculate the density of ultimate leader pairs in the initial configuration, i.e., the density of order 0. Then, we calculate an upper bound and an exact integral geometry formula for the density of ultimate leaders of order 1, or *density of order one.* Again, by ultimate leader pairs we mean a nearest neighbor pair.

3.1.1 Density of Order Zero

Consider the point process $N^{(2)}$ consisting of pairs of points of Φ which are mutually closest points. Namely

$$N^{(2)} = \sum_{i \neq j \in \mathbb{Z}} \delta_{x_i, x_j} 1_{\Phi(B(x_i \to x_j))=1}, 1_{\Phi(B(x_j \to x_i))=1},$$

where $\Phi(B(x_j \to x_i))$ denotes the number of points in the open ball centered at x_j with the radius $d(x_j, x_i)$.

The mean measure of $N^{(2)}$ is given by, for $A \subset \mathbb{R}^2 \times \mathbb{R}^2$,

$$\begin{aligned}\mathbb{E}[N^{(2)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2 \times \mathbb{R}^2} & 1_{x,y \in A} 1_{x \neq y} 1_{\Phi(B(x \to y))=1}, \\ & 1_{\Phi(B(y \to x))=1} \Phi^{(2)}(dx \times dy)],\end{aligned} \tag{3.1}$$

where $\Phi^{(2)}$ is the Poisson factorial moment measure of order 2 ([9] Section 3.3.2). By the higher order Campbell-Little-Mecke formula [6]

$$\mathbb{E}[N^{(2)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2} 1_{x,y\in A}\mathbb{E}^{x,y}[1_{\Phi(B(x\to y))=1}1_{\Phi(B(y\to x))=1}]\lambda^{(2)}(dxdy), \tag{3.2}$$

where $\mathbb{E}^{x,y}$ is the two point Palm expectation and $\lambda^{(2)}$ is the factorial Poisson moment measure of order 2 ([9] Section 3.3.2). Recall that for a Poisson point process, the n^{th} factorial moment measure equals the n^{th} power of the intensity measure. We will be using this over and over again in the calculations that follow.

Now by Slivnyak's theorem [6]

$$\mathbb{E}[N^{(2)}(A)] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{x,y\in A}\mathbb{E}[1_{\Phi'(B(x\to y))=1}1_{\Phi'(B(y\to x))=1}]\lambda^2 dxdy, \tag{3.3}$$

where $\Phi' = \Phi + \delta_x + \delta_y$. Using the change of variables $(x, y) \to (x, u)$ with $u = y - x$ and taking $A = C \times \mathbb{R}^2$, with $C \in \mathcal{B}(\mathbb{R}^2)$, we get

$$\mathbb{E}[N^{(2)}(C\times\mathbb{R}^2)] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{x\in C}1_{u\in\mathbb{R}^2}\mathbb{E}[1_{\Phi'\circ\theta_x(B(o\to u))=1}1_{\Phi'\circ\theta_x(B(u\to 0))=1}]\lambda^2 dxdu. \tag{3.4}$$

The expectation is the probability for point u to be the closest to 0 and for 0 to be closest to u. Then

$$\mathbb{E}[N^{(2)}(C\times\mathbb{R}^2)] = |C|\int_{\mathbb{R}^2} 1_{u\in\mathbb{R}^2}\mathbb{E}[1_{\Phi(B(o\to u))=0}1_{\Phi(B(u\to 0))=0}]\lambda^2 du. \tag{3.5}$$

Because of symmetry, and after switching to polar coordinates, we get:

$$\mathbb{E}[N^{(2)}(C\times\mathbb{R}^2)] = |C|\int_{\theta\in[0,2\pi]}\int_{\mathbb{R}} ve^{-\lambda v^2\pi}e^{-\lambda v^2(\pi/3+\sqrt{3}/2)}\lambda^2 dv. \tag{3.6}$$

Hence

$$\mathbb{E}[N^{(2)}(C\times\mathbb{R}^2)] = |C|\int_{\mathbb{R}} 2\pi ve^{-\lambda v^2\pi}e^{-v^2(\pi/3+\sqrt{3}/2)}\lambda^2 dv. \tag{3.7}$$

To evaluate the integral, we use the change of variables $v \to r$ with $r = \pi v^2$, and recall that $\lambda = 1$. Then we get

$$\mathbb{E}[N^{(2)}(C \times \mathbb{R}^2)]) = |C| \int_0^\infty e^{-r(\frac{\pi + (\pi/3 + \sqrt{3}/2)}{\pi})} dr$$

$$\mathbb{E}[N^{(2)}(C \times \mathbb{R}^2)]) = |C| \cdot \frac{\pi}{\pi + \pi/3 + \sqrt{3}/2} \approx 0.62|C|.$$

3.1.2 Density of Order One

In this subsection, we continue with similar calculations in order to find the intensity of ultimate leader pairs of order 1. We first give an upper bound on this frequency and then show how to adjust this to get the exact frequency. For each calculation, we need the positions of at least 4 points in order to know the positions at step 1. There are two distinct configurations in which we obtain an ultimate leader pairs at step 1. A proof of the fact that there are no other possible ways of forming ultimate leader pairs of order 1 is given in Appendix B. The first type covers the situations where two followers of one point become an ultimate leader pairs at step 1. The second type covers the situations where a leader and a follower become an ultimate leader pair at the next step. Some examples of each are shown in Figure 3.1. These are the only two ways to create ultimate leader pairs at step 1.

Type 1 As mentioned above, ultimate leader pairs of order 1, type 1 are formed from the configurations where the two step 0 followers of one agent become an ultimate leader pair at step 1. We first give an upper bound on the frequency of type 1 and then we analyze the exact frequency.

Let $\mathcal{D} \in \mathcal{B}^4$ be the set of all distinct points x_1, x_2, x_3, and x_4 that satisfy the conditions (3.8)-(3.11) defined below:

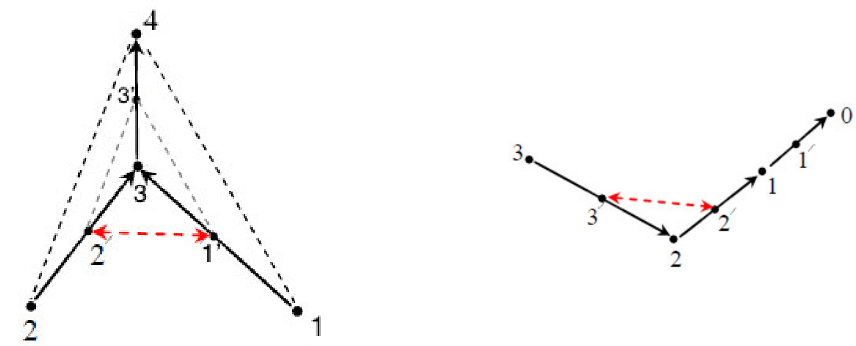


Figure 3.1: **Left:** Four points and their positions at step 1. Example of ultimate leader pair of order 1, type 1: two followers of one point become an ultimate leader pair at step 1. **Right**: Example of an ultimate leader pair of order 1, type 2. A point and its leader become an ultimate leader pair at step 1.

$$d(x_1, x_3) < d(x_1, x_2),\ d(x_1, x_3) < d(x_1, x_4), \tag{3.8}$$

are the conditions needed for x_1 to follow x_3, in the absence of other points than these four points;

$$d(x_2, x_3) < d(x_2, x_1),\ d(x_2, x_3) < d(x_2, x_4), \tag{3.9}$$

are the conditions needed for x_2 to follow x_3 as well, also in the absence of other points. Finally

$$d(x_3, x_4) < d(x_3, x_1),\ d(x_3, x_4) < d(x_3, x_2), \tag{3.10}$$

are the conditions needed for x_3 to follow x_4 under the same conditions. The phenomenon we are interested in is that where, in the above configuration,

$$d(x'_1, x'_2) < d(x'_1, x'_3),\ d(x'_2, x'_1) < d(x'_2, x'_3), \tag{3.11}$$

where $x'_1 = \frac{x_1+x_3}{2}$, $x'_2 = \frac{x_2+x_3}{2}$, and $x'_3 = \frac{x_3+x_4}{2}$. That is x'_1 and x'_2 are mutual closest neighbors in the absence of other points. Note that $\mathcal{D}$ is a semi-algebraic set.

For the points $(x_1, ..., x_4)$ to be involved in the formation of an ultimate leader pair of type 1, it is necessary but not sufficient that $(x_1, \ldots, x_4)$ belong to $\mathcal{D}$. For this to happen, in addition, it must be that there are no other points of the Poisson P.P. that change the facts that both x_1 and x_2 follow x_3, x_3 follows x_4, and x'_1 and x'_2 are mutually nearest neighbors.

Let $\Phi_{1,2,3,4}$ denote the point process Φ restricted to the set x_1, x_2, x_3, and x_4. Let $N_1^{(4)}$ be the point process of quadruples of points of the Poisson P.P. Φ that belong to $\mathcal{D}$, and are such that the following event M_1 holds: in Φ, both x_1 and x_2 follow x_3 and x_3 follows x_4.

In a first step, we evaluate the spatial frequency β^1 of the event M_1, which is an upper bound on the frequency of ultimate leader pairs of order 1, type 1.

For $A \in \mathcal{B}^4$, the mean measure of $N_1^{(4)}$ is given by,

$$\mathbb{E}[N_1^{(4)}(A)] = \mathbb{E}[\sum_{x_1,x_2,x_3,x_4 \in \Phi}^{\neq} 1_{x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}} 1_{\Phi(B(x_1 \to x_3))=1}, \\ 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1}],$$

where Φ is the P.P.P. and $B(x \to y)$ is the open ball of center x and radius $|x-y|$. In integral form, this is

$$\mathbb{E}[N_1^{(4)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} 1_{x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}} 1_{\Phi(B(x_1 \to x_3))=1}, 1_{\Phi(B(x_2 \to x_3))=1} \\ 1_{\Phi(B(x_3 \to x_4))=1} \Phi^{(4)}(dx_1 \times dx_2 \times dx_3 \times dx_4)].$$

By the higher order Campbell-Little-Mecke formula [9],

$$\mathbb{E}[N_1^{(4)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4\in A\cap\mathcal{D}}$$
$$\mathbb{E}^{x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_3))=1}, 1_{\Phi(B(x_2\to x_3))=1}1_{\Phi(B(x_3\to x_4))=1}]\lambda^{(4)}(dx_1dx_2dx_3dx_4),$$

where $\mathbb{E}^{x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x_1,...,x_4$, and $\lambda^{(4)}$ is the Poisson factorial moment measure of order 4 ([9] Section 3.3.2). By Slivnyak's theorem,

$$\mathbb{E}[N_1^{(4)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4\in A\cap\mathcal{D}}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_3))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_3\to x_4))=0}]\lambda^4 dx_1dx_2dx_3dx_4,$$

where $\hat{\Phi} = \Phi + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P of intensity λ. We can simplify the expression a bit more using stationarity of the Poisson point process. Take $A = \mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2$ with $C\in\mathcal{B}(\mathbb{R}^2)$ a compact. Because we have a stationary point process,using the change of variables, $\tilde{x}_1 = x_1 - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_1^{(4)}(\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2)] = |C|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to 0)\cup B(\tilde{x}_2\to 0))}\lambda^4 d\tilde{x}_1d\tilde{x}_2d\tilde{x}_4 = |C|\beta_1,$$

with $\tilde{\mathcal{D}} = \{(x_1 - x_3,\, x_2 - x_3,\, 0,\, x_4 - x_3) : (x_1,\, x_2,\, x_3,\, x_4)\in\mathcal{D}\}$.

This is the announced integral over a semi-algebraic set. Note that this integral is twice the frequency of interest since we do not distinguish whether $d(x_1, x_3) < d(x_2, x_3)$ or the other way around. In order to evaluate the integral, we need to write a formula for the volume of the union of 3 balls. This is discussed in Section 3.3.2.

We now discuss the exact calculation. In order to get the event of interest, one should in addition have $\Phi_1(B(x_1\to x_2)) = 1$ and $\Phi_1(B(x_2\to x_1)) = 1$, with Φ_1 the point process at step 1. In other words, certain refinements of the above configurations should be removed from the counting. We can order these refinements in the disjoint and exhaustive categories listed below

1. There is an extra point x in Φ that follows x_1 in Φ and such that the distance from x' to x'_1 is less than that from x'_1 to x'_2. The case where there is an extra point x in Φ that follows x_2 in Φ and such that the distance $d(x, x_2, \Phi_1) < d(x_1, x_2, \Phi_1)$ is analogous and is counted here as well. This is due to a fact that we do not distinguish whether $d(x_1, x_3) < d(x_2, x_3)$ or vice versa.

   For evaluating the frequency of this event, we have to enrich $\mathcal{D}$ by adding x and state that x follows x_1 and that none of the points x_1, x_2, x_3, x_4 follows x, that is

   $$\begin{aligned} d(x, x_1) < d(x, x_2),\ d(x, x_1) < d(x, x_3),\ d(x, x_1) < d(x, x_4), \\ d(x_1, x_3) < d(x_1, x),\ d(x_2, x_3) < d(x_2, x),\ d(x_4, x_3) < d(x_4, x). \end{aligned} \tag{3.12}$$

   This adds 6 quadratic inequalities. Finally, the condition that $d(x, x_1, \Phi_1) < d(x_1, x_2, \Phi_1)$, gives one more quadratic inequality

   $$d(x', x'_1) < d(x'_1, x'_2), \tag{3.13}$$

   where $x' = \frac{x+x_1}{2}$. So the frequency of this refinement can also be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_1$, which is a refinement of $\mathcal{D}$ with one more variable and 7 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

   For $A \in \mathcal{B}^5$, the mean measure of $N^{(5)}_{1,1}$, which is the point process of the

5-tuples of points satisfying the above conditions, is given by,

$$\begin{aligned}\mathbb{E}[N_{1,1}^{(5)}(A)] &= \mathbb{E}[\sum_{x,x_1,x_2,x_3,x_4\in\Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_1} 1_{\Phi(B(x_1\to x_3))=1}, \\ &\quad 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to x_1))=1}],\end{aligned}$$

In integral form, this is

$$\mathbb{E}[N_{1,1}^{(5)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_1} 1_{\Phi(B(x_1\to x_3))=1},$$

$$1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to x_1))=1} \Phi^{(5)}(dx\times dx_1\times dx_2\times dx_3\times dx_4)].$$

By the higher order Campbell-Little-Mecke formula,

$$\begin{aligned}\mathbb{E}[N_{1,1}^{(5)}(A)] &= \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_1} \\ &\quad \mathbb{E}^{x,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_3))=1}, 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} \\ &\quad 1_{\Phi(B(x\to x_1))=1}] \lambda^{(5)}(dx dx_1 dx_2 dx_3 dx_4),\end{aligned}$$

where $\mathbb{E}^{x,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, x_1, ...x_4$, and $\lambda^{(5)}$ is the factorial Poisson moment measure of order 5. By Slivnyak's theorem,

$$\begin{aligned}\mathbb{E}[N_{1,1}^{(5)}(A)] &= \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_1} \mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_3))=0}, \\ &\quad 1_{\hat{\Phi}(B(x_2\to x_3))=0} 1_{\hat{\Phi}(B(x_3\to x_4))=0} \\ &\quad 1_{\hat{\Phi}(B(x\to x_1))=1}] \lambda^5 dx dx_1 dx_2 dx_3 dx_4,\end{aligned}$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\begin{aligned}\mathbb{E}[N_{1,1}^{(5)}(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2)] &= |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}}_1} \\ &\quad e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to 0)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to\tilde{x}_1)))} \lambda^5 d\tilde{x}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,\end{aligned}$$

with $\tilde{\mathcal{D}}_1 = \{(x{-}x_3,\, x_1{-}x_3,\, x_2{-}x_3,\, 0,\, x_4{-}x_3) : (x,\, x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}_1\}$. Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

2. There is an extra point x in Φ that follows x_3 in Φ and such that the distance $d(x, x_1, \Phi_1) < d(x_1, x_2, \Phi_1)$. The case where $d(x, x_2, \Phi_1) < d(x_1, x_2, \Phi_1)$ is symmetric.

   For evaluating the frequency of this event, we have to enrich $\mathcal{D}$ by adding x and state that x follows x_3 and that none of the points x_1, x_2, x_3 follows x.

$$\begin{aligned} d(x, x_3) < d(x, x_1),\ d(x, x_3) < d(x, x_2),\ d(x, x_3) < d(x, x_4), \\ d(x_1, x_3) < d(x_1, x),\ d(x_2, x_3) < d(x_2, x),\ d(x_4, x_3) < d(x_4, x). \end{aligned} \tag{3.14}$$

   This adds 5 quadratic inequalities. Finally, the condition that $d(x, x_1, \Phi_1) < d(x_1, x_2, \Phi_1)$, gives one more quadratic inequality:

$$d(x', x_1') < d(x_1', x_2'), \tag{3.15}$$

   where we denote as $x' = \frac{x+x_1}{2}$. So the frequency of this refinement can be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_2$, which is a refinement of $\mathcal{D}$ with one more variable and 6 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

   For $A \in \mathcal{B}^5$, the mean measure of $N_{1,2}^{(5)}$, which is the point process of the 5-tuples of points satisfying the conditions above, is given by,

$$\begin{aligned} \mathbb{E}[N_{1,2}^{(5)}(A)] \;=\; \mathbb{E}[& \sum_{x,x_1,x_2,x_3,x_4 \in \Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_2} 1_{\Phi(B(x_1 \to x_3))=1}, \\ & 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} 1_{\Phi(B(x \to x_3))=1}], \end{aligned}$$

In integral form, this is

$$\mathbb{E}[N_{1,2}^{(5)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_2} 1_{\Phi(B(x_1\to x_3))=1},$$
$$1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to x_3))=1} \Phi^{(5)}(dx\times dx_1\times dx_2\times dx_3\times dx_4)].$$

By the higher order Campbell-Little-Mecke formula,

$$\mathbb{E}[N_{1,2}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_2}$$
$$\mathbb{E}^{x,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_3))=1}, 1_{\Phi(B(x_2\to x_3))=1}$$
$$1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to x_3))=1}]\lambda^{(5)}(dxdx_1dx_2dx_3dx_4),$$

where $\mathbb{E}^{x,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, x_1, ...x_4$, and $\lambda^{(5)}$ is the factorial Poisson moment measure of order 5. By Slivnyak's theorem,

$$\mathbb{E}[N_{1,2}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_0'} \mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_3))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0} 1_{\hat{\Phi}(B(x_3\to x_4))=0} 1_{\hat{\Phi}(B(x\to x_3))=1}]\lambda^5 dxdx_1dx_2dx_3dx_4,$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_{1,2}^{(5)}(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}}_2}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to 0)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to 0)))}\lambda^5 d\tilde{x}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}}_2 = \{(x-x_3,\ x_1-x_3,\ x_2-x_3,\ 0,\ x_4-x_3) : (x,\ x_1,\ x_2,\ x_3,\ x_4) \in \mathcal{D}_2\}$. Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

3. There are two extra points x and y such that x follows y (4 inequalities) and none of the points $x_1, \dots, x_3$ follows either x or y (6 inequalities). Thus we have

$$d(x,y) < d(x,x_1),\ d(x,y) < d(x,x_2),\ d(x,y) < d(x,x_3),\ d(x,y) < d(x,x_4),$$
$$d(x_1,x_3) < d(x_1,x),\ d(x_1,x_3) < d(x_1,y),$$
$$d(x_2,x_3) < d(x_2,x),\ d(x_2,x_3) < d(x_2,y),$$
$$d(x_3,x_4) < d(x_3,x),\ d(x_3,x_4) < d(x_3,y).$$

WLOG take $x' = \frac{x+y}{2}$ to be closer to x'_1 than x'_1 to x'_2, i.e.,

$$d(x'_1, x) < d(x'_1, x'_2).$$

This amounts to 2 more variables and 11 more quadratic inequalities and one more empty ball conditions ($\Phi(B(x \to y)) = 1$). So the frequency of this refinement can be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_3$, which is a refinement of $\mathcal{D}$ with two more variables and 11 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

For $A \in \mathcal{B}^6$, the mean measure of $N_{1,3}^{(6)}$, which is the point process of the 6-tuples of points satisfying the conditions above, is given by,

$$\begin{aligned}\mathbb{E}[N_{1,3}^{(6)}(A)] \;\;=\;\; \mathbb{E}[& \sum_{x,x_1,x_2,x_3,x_4 \in \Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_3} 1_{\Phi(B(x_1 \to x_3))=1}, \\ & 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} 1_{\Phi(B(x \to y))=1}],\end{aligned}$$

In integral form, this is

$$\begin{aligned}\mathbb{E}[N_{1,3}^{(6)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} & 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_3} \\ & 1_{\Phi(B(x_1 \to x_3))=1}, 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} \\ 1_{\Phi(B(x \to y))=1} \Phi^{(5)}(dx \times dy \times dx_1 \times dx_2 \times dx_3 \times dx_4)].\end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\mathbb{E}[N_{1,3}^{(6)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_3}$$
$$\mathbb{E}^{x,y,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_3))=1}, 1_{\Phi(B(x_2\to x_3))=1}$$
$$1_{\Phi(B(x_3\to x_4))=1}1_{\Phi(B(x\to y))=1}]\lambda^{(6)}(dx\,dy\,dx_1dx_2dx_3dx_4),$$

where $\mathbb{E}^{x,y,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, y, x_1, ..., x_4$, and $\lambda^{(6)}$ is the factorial Poisson moment measure of order 6. By Slivnyak's theorem,

$$\mathbb{E}[N_{1,3}^{(6)}(A\cap\mathcal{D}_3)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,y,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_0'}$$
$$\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_3))=0}, 1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_3\to x_4))=0}$$
$$1_{\hat{\Phi}(B(x\to y))=1}]\lambda^5 dx\,dy\,dx_1dx_2dx_3dx_4,$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_y + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{y} = y - x_3$ $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_{1,3}^{(6)}(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2)] = |C|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}_3}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to 0)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to\tilde{y})))}$$
$$\lambda^6 d\tilde{x}d\tilde{y}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,$$

with

$$\tilde{\mathcal{D}_3} = \{(x - x_3,\ y - x_3,\ x_1 - x_3,\ x_2 - x_3, 0, x_4 - x_3) : (x, y, x_1,\ x_2,\ x_3,\ x_4)\in\mathcal{D}_3\}.$$

Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

Let β_i the frequency of event $i = 1, 2, 3$ in the above list. Then the frequency of interest is

$$\frac{1}{2}(\beta^1 - \sum_{i=1}^{3} \beta_i). \tag{3.16}$$

Type 2 Ultimate leader pairs of order 1, type 2 represents all the configurations when a leader/follower pair forms an ultimate leader pair at the next step. Like for type 1, we give conditions for an upper bound estimate and then adjust. In a similar manner we give conditions for ultimate leader pair of order 1, type 2 (see right Figure 3.1 for an example). Let $\mathcal{D} \in \mathcal{B}^4$ be the set of all distinct points x_1, x_2, x_3, and x_4 that satisfy the following conditions:

$$x_1, x_2, x_3, x_4 \in \mathcal{B}^4 \tag{3.17}$$

$$d(x_1, x_2) < d(x_1, x_3),\ d(x_1, x_2) < d(x_1, x_4), \tag{3.18}$$

which are the conditions needed for x_1 to follow x_2, in the absence of other point;

$$d(x_2, x_3) < d(x_2, x_1),\ d(x_2, x_3) < d(x_2, x_4), \tag{3.19}$$

which are the conditions needed for x_2 to follow x_3;

$$d(x_3, x_4) < d(x_3, x_1),\ d(x_3, x_4) < d(x_3, x_2), \tag{3.20}$$

which are the conditions needed for x_3 to follow x_4, and finally in order for the ultimate leader pair of order 1 to happen, we need

$$d(x'_2, x'_1) < d(x'_2, x'_3), \tag{3.21}$$

where $x'_1 = \frac{x_1+x_2}{2}$, $x'_2 = \frac{x_2+x_3}{2}$, and $x'_3 = \frac{x_3+x_4}{2}$.

For the points $(x_1, ..., x_4)$ to be involved in the formation of an ultimate leader pair of type 2, it is necessary but not sufficient that $(x_1, \ldots, x_4)$ belong to $\mathcal{D}$. For this to happen, in addition, it must be that there are no other points of the Poisson P.P. that change the facts that x_1 follows x_2, x_2 follows x_3, x_3 follows x_4, x'_1 is closer to x'_2 than to x'_3, and x'_2 is closer to x'_1 than to x'_3.

Let $N_2^{(4)}$ be the point process of quadruples of points of the Poisson P.P. Φ that belong to $\mathcal{D}$, and are such that the following event M_2 holds: in Φ, x_1 follows x_2, x_2 follows x_3, and x_3 follows x_4 and

In a first step, we evaluate the spatial frequency β^2 of the event M_2. Recall that β^2 is an upper bound on the frequency of the ultimate leader pairs of order 1, type 2.

For $A \in \mathcal{B}^4$, the mean measure of $N_2^{(4)}$ is given by,

$$\begin{aligned}\mathbb{E}[N_1^{(4)}(A)] &= \mathbb{E}[\sum_{x_1,x_2,x_3,x_4\in\Phi}^{\neq} 1_{x_1,x_2,x_3,x_4\in A\cap\mathcal{D}} 1_{\Phi(B(x_1\to x_2))=1}, \\ &\quad 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1}],\end{aligned}$$

where Φ is the P.P.P. and $B(x \to y)$ is the open ball of center x and radius $|x-y|$. In integral form, this is

$$\begin{aligned}\mathbb{E}[N_1^{(4)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} & 1_{x_1,x_2,x_3,x_4\in A\cap\mathcal{D}} 1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1} \\ & 1_{\Phi(B(x_3\to x_4))=1} \Phi^{(4)}(dx_1 \times dx_2 \times dx_3 \times dx_4)].\end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\begin{aligned}\mathbb{E}[N_1^{(4)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} & 1_{x_1,x_2,x_3,x_4\in A\cap\mathcal{D}} \\ \mathbb{E}^{x_1,x_2,x_3,x_4}[& 1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1} \\ & 1_{\Phi(B(x_3\to x_4))=1}]\lambda^{(4)}(dx_1dx_2dx_3dx_4),\end{aligned}$$

where $\mathbb{E}^{x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x_1,...,x_4$, and $\lambda^{(4)}$ is the factorial Poisson moment measure of order 4. By Slivnyak's theorem,

$$\mathbb{E}[N_1^{(4)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4\in A\cap\mathcal{D}}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_3\to x_4))=0}]\lambda^4 dx_1dx_2dx_3dx_4,$$

where $\hat{\Phi} = \Phi + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_2$, $\tilde{x}_3 = x_3 - x_2$, and $\tilde{x}_4 = x_4 - x_2$, we get

$$\mathbb{E}[N_1^{(4)}(\mathbb{R}^2 \times C \times \mathbb{R}^2 \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}}}$$
$$e^{-\lambda(\mathrm{Vol}(B(\tilde{x}_3\to\tilde{x}_4)\cup B(\tilde{x}_1\to 0)\cup B(0\to\tilde{x}_3))}\lambda^4 d\tilde{x}_1 d\tilde{x}_3 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}} = \{(x_1 - x_2, 0, x_3 - x_2, x_4 - x_2) : (x_1, x_2, x_3, x_4) \in \mathcal{D}\}$.

In order to evaluate the integral, we need to write a formula for the volume of the union of 3 balls. Again, details are left for Section 3.3.2.

We see that as announced the upper bound frequency, β^2, of phenomenon M_2 is obtained as an integral over a semi-algebraic set.

Now that we have an upper bound frequency for the formation of an ultimate leader pair of order 1, type 2, we can refine it to get to exact values. As above, we break down into disjoint groups of conditions that factor into the calculation of the exact frequency. This analysis is same as for the ultimate leader pair of order 1, type 1.

We now discuss the exact calculation. In order to get the event of interest, need to have $\Phi_1(B(x_1 \to x_2)) = 1$ and $\Phi_1(B(x_2 \to x_1)) = 1$, with Φ_1 the point process at step 1. In other words, certain refinements of the above configurations should be removed from the counting. We can again order these refinements in the disjoint and exhaustive categories listed below:

1. There is an extra point x in Φ that follows x_1 in Φ and such that the distance from x' to x_1' is less than that from x_1' to x_2'.

   For evaluating the frequency of this event, we have to enrich $\mathcal{D}$ by adding x and state that x follows x_1 and that none of the points x_1, x_2, x_3, x_4 follows x.

$$\begin{aligned} d(x, x_1) < d(x, x_2),\ d(x, x_1) < d(x, x_3),\ d(x, x_1) < d(x, x_4), \\ d(x_1, x_3) < d(x_1, x),\ d(x_2, x_3) < d(x_2, x),\ d(x_4, x_3) < d(x_4, x). \end{aligned} \tag{3.22}$$

   This adds 6 quadratic inequalities. Finally, the condition that $d(x, x_1, \Phi_1) < d(x_1, x_2, \Phi_1)$, where $x' = \frac{x+x_1}{2}$, gives one more quadratic inequality:

$$d(x', x_1') < d(x_1', x_2'). \tag{3.23}$$

   So the frequency of this refinement can be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_1$, which is a refinement of $\mathcal{D}$ with one more variable and 7 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

   For $A \in \mathcal{B}^5$, the mean measure of $N_{2,1}^{(5)}$, which is the point process of the 5-tuples of points satisfying the conditions above, is given by,

$$\begin{aligned} \mathbb{E}[N_{2,1}^{(5)}(A)] \ = \ \mathbb{E}[& \sum_{x, x_1, x_2, x_3, x_4 \in \Phi}^{\neq} 1_{x, x_1, x_2, x_3, x_4 \in A \cap \mathcal{D}_1} 1_{\Phi(B(x_1 \to x_2))=1}, \\ & 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} 1_{\Phi(B(x \to x_1))=1}], \end{aligned}$$

   In integral form, this is

$$\begin{aligned} \mathbb{E}[N_{2,1}^{(5)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} 1_{x, x_1, x_2, x_3, x_4 \in A \cap \mathcal{D}_1} \\ 1_{\Phi(B(x_1 \to x_2))=1}, 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} 1_{\Phi(B(x \to x_1))=1} \\ \Phi^{(5)}(dx \times dx_1 \times dx_2 \times dx_3 \times dx_4)]. \end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\mathbb{E}[N_{2,1}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_1}$$
$$\mathbb{E}^{x,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1}1_{\Phi(B(x_3\to x_4))=1}$$
$$1_{\Phi(B(x\to x_1))=1}]\lambda^{(5)}(dx\, dx_1 dx_2 dx_3 dx_4),$$

where $\mathbb{E}^{x,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, x_1, ..., x_4$, and $\lambda^{(5)}$ is the factorial Poisson moment measure of order 5. By Slivnyak's theorem,

$$\mathbb{E}[N_{2,1}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_1}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_3\to x_4))=0}1_{\hat{\Phi}(B(x\to x_1))=1}]\lambda^5 dx dx_1 dx_2 dx_3 dx_4,$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_{2,1}^{(5)}(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}_1}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to\tilde{x}_2)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to\tilde{x}_1)))}\lambda^5 d\tilde{x}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}_1} = \{(x-x_3,\, x_1-x_3,\, x_2-x_3,\, 0,\, x_4-x_3) : (x,\, x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}_1\}$. Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

2. There is an extra point x in Φ that follows x_2 in Φ and such that the distance $d(x, x_1, \Phi_1) < d(x_1, x_2, \Phi_1)$.

   For evaluating the frequency of this event, we have to enrich $\mathcal{D}$ by adding x and state that x follows x_3 and that none of the points x_1, x_2, x_3 follows

x. That is

$$\begin{aligned} d(x, x_2) < d(x, x_1),\ d(x, x_2) < d(x, x_3),\ d(x, x_2) < d(x, x_4), \\ d(x_1, x_2) < d(x_1, x),\ d(x_2, x_3) < d(x_2, x),\ d(x_4, x_3) < d(x_4, x). \end{aligned} \tag{3.24}$$

This adds 5 quadratic inequalities. Finally, the condition that $d(x, x_1, \Phi_1) < d(x_1, x_2, \Phi_1)$, gives one more quadratic inequality:

$$d(x', x'_1) < d(x'_1, x'_2), \tag{3.25}$$

where $x' = \frac{x+x_1}{2}$. So the frequency of this refinement can also be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_2$, which is a refinement of $\mathcal{D}$ with one more variable and 6 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

These conditions are actually equivalent to the setup of the type 2 follower inversion (Section 3.2.2).

For $A \in \mathcal{B}^5$, the mean measure of $N^{(5)}_{2,2}$, which is the point process of the 5-tuples of points satisfying the conditions above, is given by,

$$\begin{aligned} \mathbb{E}[N^{(5)}_{2,2}(A)] \;\; = \;\; & \mathbb{E}[\sum_{x,x_1,x_2,x_3,x_4 \in \Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_2} 1_{\Phi(B(x_1 \to x_2))=1}, \\ & 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} 1_{\Phi(B(x \to x_2))=1}]. \end{aligned}$$

In integral form, this is

$$\begin{aligned} \mathbb{E}[N^{(5)}_{2,2}(A)] = \mathbb{E}[\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_2} \\ 1_{\Phi(B(x_1 \to x_2))=1}, 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} \\ 1_{\Phi(B(x \to x_2))=1} \Phi^{(5)}(dx \times dx_1 \times dx_2 \times dx_3 \times dx_4)]. \end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\mathbb{E}[N_{2,2}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_2}$$
$$\mathbb{E}^{x,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1}1_{\Phi(B(x_3\to x_4))=1}$$
$$1_{\Phi(B(x\to x_2))=1}]\lambda^{(5)}(dx\, dx_1 dx_2 dx_3 dx_4),$$

where $\mathbb{E}^{x,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, x_1, ..., x_4$, and $\lambda^{(5)}$ is the factorial Poisson moment measure of order 5. By Slivnyak's theorem,

$$\mathbb{E}[N_{2,2}^{(5)}(A] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_2}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_3\to x_4))=0}1_{\hat{\Phi}(B(x\to x_2))=1}]\lambda^5 dx dx_1 dx_2 dx_3 dx_4,$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_{2,2}^{(5)}(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}_2}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to\tilde{x}_2)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to\tilde{x}_2)))}\lambda^5 d\tilde{x}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}_2} = \{(x-x_3,\, x_1-x_3,\, x_2-x_3,\, 0,\, x_4-x_3) : (x,\, x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}_2\}$. Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

3. There is an extra point x in Φ that follows x_2 in Φ and such that the distance $d(x, x_2, \Phi_1) < d(x_1, x_2, \Phi_1)$.

   For evaluating the frequency of this event, we have to enrich $\mathcal{D}$ by adding x and state that x follows x_2 and that none of the points x_1, x_2, x_3 follows

x.

$$\begin{aligned} d(x,x_2) < d(x,x_1),\ d(x,x_2) < d(x,x_3),\ d(x,x_2) < d(x,x_4), \\ d(x_1,x_2) < d(x_1,x),\ d(x_2,x_3) < d(x_2,x),\ d(x_4,x_3) < d(x_4,x). \end{aligned} \tag{3.26}$$

This adds 5 quadratic inequalities. Finally, the condition that $d(x, x_2, \Phi_1) < d(x_1, x_2, \Phi_1)$, gives one more quadratic inequality:

$$d(x', x_2') < d(x_1', x_2'), \tag{3.27}$$

where $x' = \frac{x+x_1}{2}$. So the frequency of this refinement can also be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_3$, which is a refinement of $\mathcal{D}$ with one more variable and 6 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

For $A \in \mathcal{B}^5$, the mean measure of $N_{2,3}^{(5)}$, which is the point process of the 5-tuples of points satisfying the conditions above, is given by,

$$\begin{aligned} \mathbb{E}[N_{2,3}^{(5)}(A)] &= \mathbb{E}[\sum_{x,x_1,x_2,x_3,x_4 \in \Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_3} 1_{\Phi(B(x_1 \to x_2))=1}, \\ &\quad 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} 1_{\Phi(B(x \to x_2))=1}], \end{aligned}$$

In integral form, this is

$$\begin{aligned} \mathbb{E}[N_{2,3}^{(5)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} & 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_3} 1_{\Phi(B(x_1 \to x_2))=1}, \\ & 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} \\ & 1_{\Phi(B(x \to x_2))=1} \Phi^{(5)}(dx \times dx_1 \times dx_2 \times dx_3 \times dx_4)]. \end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\begin{aligned} \mathbb{E}[N_{2,3}^{(5)}(A)] = \int_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} & 1_{x,x_1,x_2,x_3,x_4 \in A \cap \mathcal{D}_3} \\ \mathbb{E}^{x,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1 \to x_2))=1}, & 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} \\ & 1_{\Phi(B(x \to x_2))=1}] \lambda^{(5)}(dx\, dx_1 dx_2 dx_3 dx_4), \end{aligned}$$

where $\mathbb{E}^{x,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, x_1, ..., x_4$, and $\lambda^{(5)}$ is the factorial Poisson moment measure of order 5. By Slivnyak's theorem,

$$\mathbb{E}[N_{2,3}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_3}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_3\to x_4))=0}1_{\hat{\Phi}(B(x\to x_2))=1}]\lambda^5 dxdx_1dx_2dx_3dx_4,$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_{2,3}^{(5)}(\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2)] = |C|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4\in\tilde{\mathcal{D}}_3}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to\tilde{x}_2)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to\tilde{x}_2)))}\lambda^5 d\tilde{x}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}_3} = \{(x-x_3,\, x_1-x_3,\, x_2-x_3,\, 0,\, x_4-x_3) : (x,\, x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}_3\}$.

4. There is an extra point x in Φ that follows x_3 in Φ and such that the distance $d(x, x_2, \Phi_1) < d(x_1, x_2, \Phi_1)$.

   For evaluating the frequency of this event, we have to enrich $\mathcal{D}$ by adding x and state that x follows x_3 and that none of the points x_1, x_2, x_3 follow x.

$$\begin{aligned} d(x,x_3) < d(x,x_1),\ d(x,x_3) < d(x,x_2),\ d(x,x_3) < d(x,x_4),\\ d(x_1,x_2) < d(x_1,x),\ d(x_2,x_3) < d(x_2,x),\ d(x_4,x_3) < d(x_4,x). \end{aligned} \tag{3.28}$$

   This adds 5 quadratic inequalities. Finally, the condition that $d(x, x_2, \Phi_1) < d(x_1, x_2, \Phi_1)$, gives one more quadratic inequality:

$$d(x', x_2') < d(x_1', x_2'), \tag{3.29}$$

   where we denote as $x' = \frac{x+x_1}{2}$. So the frequency of this refinement can be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_4$, which is a refinement of $\mathcal{D}$ with one more variable and 6 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

For $A \in \mathcal{B}^5$, the mean measure of $N_{2,4}^{(5)}$, which is the point process of the 5-tuples of points satisfying the conditions above, is given by,

$$\begin{aligned}\mathbb{E}[N_{2,4}^{(5)}(A)] &= \mathbb{E}[\sum_{x,x_1,x_2,x_3,x_4\in\Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_4} 1_{\Phi(B(x_1\to x_2))=1},\\ &\quad 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to x_3))=1}],\end{aligned}$$

In integral form, this is

$$\begin{aligned}\mathbb{E}[N_{2,4}^{(5)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_4}\\ 1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1}\\ 1_{\Phi(B(x\to x_3))=1}\Phi^{(5)}(dx\times dx_1\times dx_2\times dx_3\times dx_4)].\end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\begin{aligned}\mathbb{E}[N_{2,4}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_4}\\ \mathbb{E}^{x,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1}\\ 1_{\Phi(B(x\to x_3))=1}]\lambda^{(5)}(dx\, dx_1 dx_2 dx_3 dx_4),\end{aligned}$$

where $\mathbb{E}^{x,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, x_1, ..., x_4$, and $\lambda^{(5)}$ is the factorial Poisson moment measure of order 5. By Slivnyak's theorem,

$$\begin{aligned}\mathbb{E}[N_{2,4}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_4}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},\\ 1_{\hat{\Phi}(B(x_2\to x_3))=0} 1_{\hat{\Phi}(B(x_3\to x_4))=0} 1_{\hat{\Phi}(B(x\to x_3))=1}]\lambda^5 dx dx_1 dx_2 dx_3 dx_4,\end{aligned}$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N^{(5)}_{2,4}(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4 \in \tilde{\mathcal{D}}_4}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to\tilde{x}_2)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to 0)))}\lambda^5 d\tilde{x}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}}_4 = \{(x-x_3,\, x_1-x_3,\, x_2-x_3,\, 0,\, x_4-x_3) : (x,\, x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}_4\}$.

5. There are two extra points x and y such that x follows y (4 inequalities) and none of the points $x_1, \ldots, x_3$ follows either x or y (6 inequalities). Thus we have

$$d(x,y) < d(x,x_1),\ d(x,y) < d(x,x_2),\ d(x,y) < d(x,x_3),\ d(x,y) < d(x,x_4),$$
$$d(x_1,x_2) < d(x_1,x),\ d(x_1,x_2) < d(x_1,y),$$
$$d(x_2,x_3) < d(x_2,x),\ d(x_2,x_3) < d(x_2,y),$$
$$d(x_3,x_4) < d(x_3,x),\ d(x_3,x_4) < d(x_3,y).$$

Take $x' = \frac{x+y}{2}$ to be closer to x'_1 than x'_1 to x'_2, i.e.,

$$d(x'_1, x) < d(x'_1, x'_2).$$

This amounts to 2 more variables and 11 more quadratic inequalities and one more empty ball conditions ($\Phi(B(x \to y)) = 1$). So the frequency of this refinement can be reduced to the evaluation of an integral over a semi-algebraic set $\mathcal{D}_5$, which is a refinement of $\mathcal{D}$ with two more variables and 11 more quadratic inequalities, with the function to be integrated involving one more ball in the union.

For $A \in \mathcal{B}^6$, the mean measure of $N_{2,5}^{(6)}$, which is the point process of the 6-tuples of points satisfying the conditions above, is given by,

$$\begin{aligned}\mathbb{E}[N_{2,5}^{(6)}(A)] &= \mathbb{E}[\sum_{x,x_1,x_2,x_3,x_4\in\Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_5} 1_{\Phi(B(x_1\to x_2))=1}, \\ &\quad 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to y))=1}],\end{aligned}$$

In integral form, this is

$$\begin{aligned}\mathbb{E}[N_{2,5}^{(6)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_5} \\ 1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} \\ 1_{\Phi(B(x\to y))=1}\Phi^{(5)}(dx\times dy\times dx_1\times dx_2\times dx_3\times dx_4)].\end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\begin{aligned}\mathbb{E}[N_{2,5}^{(6)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_5} \\ \mathbb{E}^{x,y,x_1,x_2,x_3,x_4}[1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} \\ 1_{\Phi(B(x\to y))=1}]\lambda^{(6)}(dx\, dy\, dx_1 dx_2 dx_3 dx_4),\end{aligned}$$

where $\mathbb{E}^{x,y,x_1,x_2,x_3,x_4}$ is the Palm expectation of Φ at $x, y, x_1, ..., x_4$, and $\lambda^{(6)}$ is the factorial Poisson moment measure of order 6. By Slivnyak's theorem,

$$\begin{aligned}\mathbb{E}[N_{2,5}^{(6)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x,y,x_1,x_2,x_3,x_4\in A\cap\mathcal{D}_5}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0}, \\ 1_{\hat{\Phi}(B(x_2\to x_3))=0} 1_{\hat{\Phi}(B(x_3\to x_4))=0} 1_{\hat{\Phi}(B(x\to y))=1}]\lambda^5 dxdydx_1dx_2dx_3dx_4,\end{aligned}$$

where $\hat{\Phi} = \Phi + \delta_x + \delta_y + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4}$ is a Poisson P.P with intensity λ. Take $A = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables

$\tilde{x}_1 = x_1 - x_3$, $\tilde{x} = x - x_3$, $\tilde{y} = y - x_3$ $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_{2,5}^{(6)}(\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1_{\tilde{x},\tilde{x}_1,\tilde{x}_2,\tilde{x}_4 \in \tilde{\mathcal{D}_5}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to\tilde{x}_2)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}\to\tilde{y})))}\lambda^6 d\tilde{x}d\tilde{y}\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,$$

with

$$\tilde{\mathcal{D}_5} = \{(x-x_3, y-x_3, x_1-x_3, x_2-x_3, 0, x_4-x_3) : (x, y, x_1, x_2, x_3, x_4) \in \mathcal{D}_5\}.$$

Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

Let γ_i the frequency of event $i = 1, ..., 5$ in the above list. Then the frequency of interest is

$$\beta^2 - \sum_{i=1}^{5} \gamma_i. \tag{3.30}$$

3.2 Other Integral Geometry Frequency Estimates

Using the method for calculating frequencies described above, we calculate the frequencies of other events. The reader might concentrate on the setting and skip some repetitive derivations. First, we describe the conditions for the 4 body swap, and calculate the frequency at which it occurs. Then we describe how to calculate the frequency of follower inversions.

3.2.1 Frequency of 4 body swaps

In this section we want to look at the potential *party fusion*, and thus need the positions of at least 6 points like in Figure 3.2. We can also write an integral formula for this situation. Note that our conditions will not exclude the possibility of 6 points belonging to the same party. Let $\mathcal{D} \in \mathcal{B}^6$ be the set of all distinct points x_1, x_2, x_3, x_4, x_5, and x_6 that satisfy the following conditions:

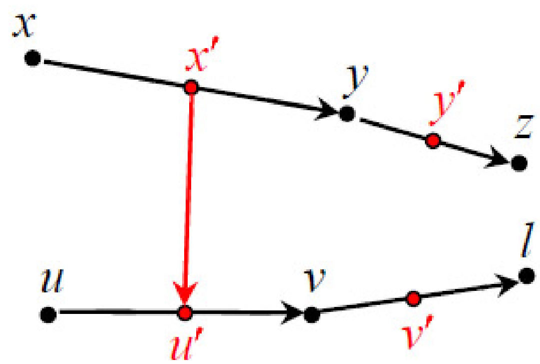


Figure 3.2: General 4 body swap with initial 6 points in black and points at step 1 in red.

$$x_1, x_2, x_3, x_4, x_5, x_6 \in \mathcal{B}^6 \tag{3.31}$$

$$\begin{aligned} &d(x_1, x_2) < d(x_1, x_3),\ d(x_1, x_2) < d(x_1, x_4), \\ &d(x_1, x_2) < d(x_1, x_5),\ d(x_1, x_2) < d(x_1, x_6), \end{aligned} \tag{3.32}$$

which are the conditions needed for x_1 to follow x_2, in the absence of other point;

$$\begin{aligned} &d(x_2, x_3) < d(x_2, x_1),\ d(x_2, x_3) < d(x_2, x_4), \\ &d(x_2, x_3) < d(x_2, x_5),\ d(x_2, x_3) < d(x_2, x_6), \end{aligned} \tag{3.33}$$

which are the conditions needed for x_2 to follow x_3;

$$\begin{aligned} &d(x_5, x_6) < d(x_5, x_1),\ d(x_5, x_6) < d(x_5, x_2), \\ &d(x_5, x_6) < d(x_5, x_3),\ d(x_5, x_2) < d(x_5, x_4), \end{aligned} \tag{3.34}$$

which are the conditions that ensure that x_5 follows x_6 in the absence of other points. Finally in order for the 4 body swap to happen, we need

$$d(x_1', x_4') < d(x_1', x_2'),\ d(x_1', x_4') < d(x_1', x_5'), \tag{3.35}$$

where $x_1' = \frac{x_1+x_2}{2}$, $x_2' = \frac{x_2+x_3}{2}$, $x_4' = \frac{x_4+x_5}{2}$, and $x_5' = \frac{x_5+x_6}{2}$.

For the points $(x_1, ..., x_6)$ to be involved in the formation of a 4 body swap, it is necessary but not sufficient that $(x_1, \ldots, x_6)$ belong to $\mathcal{D}$. For this to happen, in addition, it must be that there are no other points of the Poisson P.P. that change the facts that both x_1 and follow x_2, x_2 follows x_3, x_4 follows x_5, x_5 follows x_6, x_1' is closer to x_4' than to x_2', and x_1' is closer to x_4' than to x_5'.

Let $N_{3,2}^{(6)}$ be the point process of 6-tuples of points of the Poisson P.P. Φ that belong to $\mathcal{D}$, and are such that the following event M_3 holds: in Φ, x_1 follows x_2, x_2 follows x_3, x_4 follows x_5, and x_5 follows x_6.

In a first step, we evaluate the spatial frequency β^3 of the event M_3. Recall that β^3 is an upper bound on the frequency of the 4 body swap.

For $A \in \mathcal{B}^6$, the mean measure of $N_{3,2}^{(6)}$ is given by,

$$\begin{aligned}\mathbb{E}[N_{3,2}^{(6)}(A)] \quad = \quad & \mathbb{E}[\sum_{x_1,x_2,x_3,x_4,x_5,x_6\in\Phi}^{\neq} 1_{x_1,x_2,x_3,x_4,x_5,x_6\in A\cap\mathcal{D}} 1_{\Phi(B(x_1\to x_2))=1}, \\ & 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_4\to x_5))=1} 1_{\Phi(B(x_5\to x_6))=1}],\end{aligned}$$

In integral form, this is

$$\begin{aligned}\mathbb{E}[N_{3,2}^{(6)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} & 1_{x_1,x_2,x_3,x_4,x_5,x_6\in A\cap\mathcal{D}} 1_{\Phi(B(x_1\to x_2))=1}, \\ & 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_4\to x_5))=1} \\ & 1_{\Phi(B(x_5\to x_6))=1} \Phi^{(6)}(dx_1 \times dx_2 \times dx_3 \times dx_4 \times dx_5 \times dx_6)].\end{aligned}$$

By the higher order Campbell-Little-Mecke formula,

$$\mathbb{E}[N_{3,2}^{(6)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5,x_6\in A\cap\mathcal{D}}$$
$$\mathbb{E}^{x_1,x_2,x_3,x_4,x_5,x_6}[1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1}$$
$$1_{\Phi(B(x_4\to x_5))=1}1_{\Phi(B(x_5\to x_6))=1}]\lambda^{(6)}(dx_1dx_2dx_3dx_4dx_5dx_6),$$

where $\mathbb{E}^{x_1,x_2,x_3,x_4,x_5,x_6}$ is the Palm expectation of Φ at $x_1, ..., x_6$, and $\lambda^{(6)}$ is the factorial Poisson moment measure of order 6. By Slivnyak's theorem,

$$\mathbb{E}[N_{3,2}^{(6)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5.x_6\in A\cap\mathcal{D}_3}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_4\to x_5))=0}1_{\hat{\Phi}(B(x_5\to x_6))=1}]\lambda^6 dx_1dx_2dx_3dx_4dx_5dx_6,$$

where $\hat{\Phi} = \Phi + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4} + \delta_{x_5} + \delta_{x_6}$ with Φ a Poisson P.P. Take $A = \mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x}_5 = x_5 - x_3$, $\tilde{x}_2 = x_2 - x_3$, $\tilde{x}_4 = x_4 - x_3$, and $\tilde{x}_6 = x_6 - x_3$ we get

$$\mathbb{E}[N_{1,2}^{(6)}(\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2)] = |C|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x}_1,\tilde{x}_2,\tilde{x}_4,\tilde{x}_5,\tilde{x}_6\in\tilde{\mathcal{D}}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to\tilde{x}_2)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}_4\to\tilde{x}_5)\cup(B(\tilde{x}_5\to\tilde{x}_6)))}\lambda^6 d\tilde{x}_1d\tilde{x}_2d\tilde{x}_4d\tilde{x}_5d\tilde{x}_6,$$

with $\tilde{\mathcal{D}} = \{(x_1 - x_3,\, x_2 - x_3,\, 0,\, x_4 - x_3) : (x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}\}$.

Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2. In a similar manner like in Section 3.1.2 we can write a set of conditions to adjust the frequency β^3. Since the analysis of getting to the exact frequency is the same as in the Section 3.1.2, we omit the additional analysis here.

3.2.2 Frequency of follower inversions

There are two mutually exclusive sets of conditions that satisfy the follower inversion. Like before we split them into Type 1 and Type 2 and calculate the frequency of each separately. Examples of each are shown in the left Figure 3.3 and the right Figure 3.3 respectively. Type 2 is really a special case of type 1 where $x_2 = x_5$. However in the analysis, we take distinct points and we would not count type 2 inversion cases when calculating type 1.

Figure 3.3: **Left:** Type 1 follower inversion. **Right:** Type 2 follower inversion.

Type 1 Type 1 inversion represents all the configurations when a leader starts following its follower at the next step, and the follower does a leader switch like in the left Figure 3.3. Like above, we first give an upper bound and then adjust. Let $\mathcal{D}_4 \in \mathcal{B}^6$ be the set of all distinct points x_1, x_2, x_3, x_4, x_5, and x_6 that satisfy the following conditions:

$$x_1, x_2, x_3, x_4, x_5, x_6 \in \mathcal{B}^6 \tag{3.36}$$

$$\begin{aligned} &d(x_1, x_2) < d(x_1, x_3),\ d(x_1, x_2) < d(x_1, x_4), \\ &d(x_1, x_2) < d(x_1, x_5),\ d(x_1, x_2) < d(x_1, x_6), \end{aligned} \tag{3.37}$$

which are the conditions needed for x_1 to follow x_2, in the absence of other point;

$$\begin{aligned} &d(x_2, x_3) < d(x_2, x_1),\ d(x_2, x_3) < d(x_2, x_4), \\ &d(x_2, x_3) < d(x_2, x_5),\ d(x_2, x_3) < d(x_2, x_6), \end{aligned} \tag{3.38}$$

which are the conditions needed for x_2 to follow x_3;

$$\begin{aligned} &d(x_3, x_4) < d(x_3, x_1),\ d(x_3, x_4) < d(x_3, x_2), \\ &d(x_3, x_4) < d(x_3, x_5),\ d(x_3, x_4) < d(x_3, x_6), \end{aligned} \tag{3.39}$$

which are the conditions needed for x_3 to follow x_4, and

$$\begin{aligned} &d(x_5,x_6) < d(x_5,x_1),\ d(x_5,x_6) < d(x_5,x_2), \\ &d(x_5,x_6) < d(x_5,x_3),\ d(x_5,x_2) < d(x_5,x_4), \end{aligned} \tag{3.40}$$

which are the conditions that ensure that x_5 follows x_6 in the absence of other points. Finally in order for the type 2 inversion to happen, we need

$$d(x_2',x_1') < d(x_2',x_3'),\ d(x_1',x_5') < d(x_1',x_2'), \tag{3.41}$$

where $x_2' = \frac{x_1+x_2}{2}$, $x_3' = \frac{x_2+x_3}{2}$, $x_4' = \frac{x_3+x_4}{2}$, and $x_5' = \frac{x_5+x_6}{2}$.

For the points $(x_1, ..., x_6)$ to be involved in the formation of a type 1 inversion, it is necessary but not sufficient that $(x_1, \ldots, x_6)$ belong to $\mathcal{D}_1$. For this to happen, in addition, it must be that there are no other points of the Poisson P.P. that change the facts that both x_1 and follow x_3, x_3 follows x_4, x_5 follows x_6, x_1' is closer to x_5' than to x_2', and x_2' is closer to x_1' than to x_3'.

Let $N_{1,2}^{(6)}$ be the point process of 6-tuples of points of the Poisson P.P. Φ that belong to $\mathcal{D}_1$, and are such that the following event M_4 holds: in Φ, x_2 follows x_1, x_3 follows x_2, x_3 follows x_4, and x_5 follows x_6.

In a first step, we evaluate the spatial frequency β^4 of the event M_4. Recall that β^4 is an upper bound on the frequency of the type 1 inversion.

For $A \in \mathcal{B}^6$, the mean measure of $N_{1,2}^{(6)}$ is given by,

$$\begin{aligned} \mathbb{E}[N_{1,2}^{(6)}(A)] &= \mathbb{E}[\sum_{x_1,x_2,x_3,x_4,x_5,x_6 \in \Phi}^{\neq} 1_{x_1,x_2,x_3,x_4,x_5,x_6 \in A \cap \mathcal{D}_1} 1_{\Phi(B(x_1 \to x_2))=1}, \\ &\quad 1_{\Phi(B(x_2 \to x_3))=1} 1_{\Phi(B(x_3 \to x_4))=1} 1_{\Phi(B(x_5 \to x_6))=1}], \end{aligned}$$

In integral form, this is

$$\mathbb{E}[N_{1,2}^{(6)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5,x_6\in A\cap\mathcal{D}_1} 1_{\Phi(B(x_1\to x_2))=1},$$
$$1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1}$$
$$1_{\Phi(B(x_5\to x_6))=1}\Phi^{(6)}(dx_1\times dx_2\times dx_3\times dx_4\times dx_5\times dx_6)].$$

By the higher order Campbell-Little-Mecke formula,

$$\mathbb{E}[N_{1,2}^{(6)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5,x_6\in A\cap\mathcal{D}_1}$$
$$\mathbb{E}^{x_1,x_2,x_3,x_4,x_5,x_6}[1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1}$$
$$1_{\Phi(B(x_3\to x_4))=1}1_{\Phi(B(x_5\to x_6))=1}]\lambda^{(6)}(dx_1dx_2dx_3dx_4dx_5dx_6),$$

where $\mathbb{E}^{x_1,x_2,x_3,x_4,x_5,x_6}$ is the Palm expectation of Φ at $x_1,...,x_6$, and $\lambda^{(6)}$ is the factorial Poisson moment measure of order 6. By Slivnyak's theorem,

$$\mathbb{E}[N_{1,2}^{(6)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5.x_6\in A\cap\mathcal{D}_1}\mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0}1_{\hat{\Phi}(B(x_3\to x_4))=0}1_{\hat{\Phi}(B(x_5\to x_6))=1}]\lambda^6 dx_1dx_2dx_3dx_4dx_5dx_6,$$

where $\hat{\Phi} = \Phi + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4} + \delta_{x_5} + \delta_{x_6}$ with Φ is a Poisson P.P. Take $A = \mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x}_5 = x_5 - x_3$, $\tilde{x}_2 = x_2 - x_3$, $\tilde{x}_4 = x_4 - x_3$, and $\tilde{x}_6 = x_6 - x_3$ we get

$$\mathbb{E}[N_{1,2}^{(6)}(\mathbb{R}^2\times\mathbb{R}^2\times C\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2)] = |C|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x}_1,\tilde{x}_2,\tilde{x}_4,\tilde{x}_5,\tilde{x}_6\in\tilde{\mathcal{D}_1}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to\tilde{x}_2)\cup B(\tilde{x}_2\to 0)\cup B(\tilde{x}_5\to\tilde{x}_6)))}\lambda^6 d\tilde{x}_1d\tilde{x}_2d\tilde{x}_4d\tilde{x}_5d\tilde{x}_6,$$

with

$$\tilde{\mathcal{D}_1} = \{(x_1-x_3,\ x_2-x_3, 0,\ x_4-x_3,\ x_5-x_3,\ x_6-x_3) : (x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6)\in\mathcal{D}_1\}.$$

Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

Type 2 Type 2 inversions represent all the configurations when a leader starts following its follower at the next step, and the follower does a leader swap. Like for type 1, we give conditions for an upper bound estimate and then adjust. In a similar manner we give conditions for type 2 inversion illustrated in the right of Figure 3.3. Let $\mathcal{D}_2 \in \mathcal{B}^5$ be the set of all distinct points x_1, x_2, x_3, x_4, and x_5 that satisfy the following conditions:

$$x_1, x_2, x_3, x_4, x_5 \in \mathcal{B}^5 \tag{3.42}$$

$$d(x_1, x_2) < d(x_1, x_3),\ d(x_1, x_2) < d(x_1, x_4),\ d(x_1, x_2) < d(x_1, x_5), \tag{3.43}$$

which are the conditions needed for x_1 to follow x_2, in the absence of other point;

$$d(x_2, x_3) < d(x_2, x_1),\ d(x_2, x_3) < d(x_2, x_4),\ d(x_2, x_3) < d(x_2, x_5), \tag{3.44}$$

which are the conditions needed for x_2 to follow x_3;

$$d(x_3, x_4) < d(x_3, x_1),\ d(x_3, x_4) < d(x_3, x_2),\ d(x_3, x_4) < d(x_3, x_5), \tag{3.45}$$

which are the conditions needed for x_3 to follow x_4, and

$$d(x_5, x_2) < d(x_5, x_1),\ d(x_5, x_2) < d(x_5, x_3)\ d(x_5, x_2) < d(x_5, x_4), \tag{3.46}$$

which is the condition that ensures that x_5 follows x_2 in the absence of other points. Finally in order for the type 2 inversion to happen, we need

$$d(x_2', x_1') < d(x_2', x_3'),\ d(x_1', x_5') < d(x_1', x_2'), \tag{3.47}$$

where $x_2' = \frac{x_1+x_2}{2}$, $x_3' = \frac{x_2+x_3}{2}$, $x_4' = \frac{x_3+x_4}{2}$, and $x_5' = \frac{x_5+x_2}{2}$.

For the points $(x_1, ..., x_5)$ to be involved in the formation of a type 2 inversion, it is necessary but not sufficient that $(x_1, \ldots, x_5)$ belong to $\mathcal{D}_2$. For this to happen, in addition, it must be that there are no other points of the Poisson P.P. that change the facts that both x_1 and x_5 follow x_2, x_2 follows x_3, x_3 follows x_4, x_1' is closer to x_5' than to x_2', and x_2' is closer to x_1' than to x_3'.

Let $N_{2,2}^{(5)}$ be the point process of 5-tuple of points of the Poisson P.P. Φ that belong to $\mathcal{D}_2$, and are such that the following event M_5 holds: in Φ, x_1 follows x_2, x_2 follows x_3, x_3 follows x_4, and x_5 follows x_2.

In a first step, we evaluate the spatial frequency β^5 of the event M_5. Recall that β^5 is an upper bound on the frequency of the type 2 inversion.

For $A \in \mathcal{B}^5$, the mean measure of $N_{2,2}^{(5)}$ is given by,

$$\begin{aligned}\mathbb{E}[N_{2,2}^{(5)}(A)] &= \mathbb{E}[\sum_{x_1,x_2,x_3,x_4,x_5\in\Phi}^{\neq} 1_{x,x_1,x_2,x_3,x_4,x_5\in A\cap\mathcal{D}_2} 1_{\Phi(B(x_1\to x_2))=1}, \\ &\quad 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to x_2))=1}],\end{aligned}$$

In integral form, this is

$$\mathbb{E}[N_{2,2}^{(5)}(A)] = \mathbb{E}[\int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5\in A\cap\mathcal{D}_2} 1_{\Phi(B(x_1\to x_2))=1},$$
$$1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1} 1_{\Phi(B(x\to x_2))=1} \Phi^{(5)}(dx_1\times dx_2\times dx_3\times dx_4\times dx_5)].$$

By the higher order Campbell-Little-Mecke formula,

$$\mathbb{E}[N_{2,2}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5\in A\cap\mathcal{D}_2}$$
$$\mathbb{E}^{x_1,x_2,x_3,x_4,x_5}[1_{\Phi(B(x_1\to x_2))=1}, 1_{\Phi(B(x_2\to x_3))=1} 1_{\Phi(B(x_3\to x_4))=1}$$
$$1_{\Phi(B(x_5\to x_2))=1} 1_{x_2'\notin B(x_1'\to x_5')} 1_{x_3'\notin B(x_2'\to x_1')} 1_{x_3'\notin B(x_1'\to x_2')}] \lambda^{(5)}(dx_1 dx_2 dx_3 dx_4 dx_5),$$

where $\mathbb{E}^{x_1,x_2,x_3,x_4,x_5}$ is the Palm expectation of Φ at $x_1, ..., x_5$, and $\lambda^{(5)}$ is the factorial Poisson moment measure of order 5. By Slivnyak's theorem,

$$\mathbb{E}[N_{2,2}^{(5)}(A)] = \int_{\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2} 1_{x_1,x_2,x_3,x_4,x_5\in A\cap\mathcal{D}_2} \mathbb{E}[1_{\hat{\Phi}(B(x_1\to x_2))=0},$$
$$1_{\hat{\Phi}(B(x_2\to x_3))=0} 1_{\hat{\Phi}(B(x_3\to x_4))=0} 1_{\hat{\Phi}(B(x_5\to x_2))=1}] \lambda^5 dx_1 dx_2 dx_3 dx_4 dx_5,$$

where $\hat{\Phi} = \Phi + \delta_{x_1} + \delta_{x_2} + \delta_{x_3} + \delta_{x_4} + \delta_{x_5}$ with Φ a Poisson P.P. Take $A = \mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2 \times \mathbb{R}^2$ with C a compact. Because we have a stationary point process, using the change of variables $\tilde{x}_1 = x_1 - x_3$, $\tilde{x}_5 = x_5 - x_3$, $\tilde{x}_2 = x_2 - x_3$, and $\tilde{x}_4 = x_4 - x_3$, we get

$$\mathbb{E}[N_{2,2}^{(5)}(\mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2 \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1_{\tilde{x}_1, \tilde{x}_2, \tilde{x}_4, \tilde{x}_5 \in \tilde{\mathcal{D}_2}}$$
$$e^{-\lambda(\mathrm{Vol}(B(0 \to \tilde{x}_4) \cup B(\tilde{x}_1 \to \tilde{x}_2) \cup B(\tilde{x}_2 \to 0) \cup B(\tilde{x}_5 \to \tilde{x}_2)))} \lambda^5 d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4 d\tilde{x}_5,$$

with $\tilde{\mathcal{D}_2} = \{(x_1 - x_3,\, x_2 - x_3,\, 0,\, x_4 - x_3,\, x_5 - x_3) : (x_1,\, x_2,\, x_3,\, x_4,\, x_5) \in \mathcal{D}_2\}$.

Similar to the derivation for the union of 3 balls, we can calculate the volume of the union of 4 balls. More details can be found in Section 3.3.2.

Again for both cases we can calculate exact frequencies but we omit it here.

3.3 Numerical Estimation Methods

We apply two different methodologies in order to estimate our calculations of frequencies. The first is *discrete event spatial simulation*, which is a method that relies on sampling point processes and leveraging the ergodic properties of the Poisson point process, and factors of such point processes [5, 9]. The foundation for this approach comes from the ergodic theory of point processes theory [5]. The second method is based on the numerical evaluation of integrals, and draws from the semi-algebraic sets characterized in the discussion above. Below we explain both methods in more detail and give estimated values for the frequencies calculated above. For each method, we will first give some theoretical background and then explain how we use it in our evaluation. For simulation, we provide 95% confidence intervals for all frequencies discussed in the previous sections. In the end we compare the results from the two methods.

3.3.1 Discrete Event Spatial Simulation

The theoretical background about ergodicity discussed in this section comes from [9].

Definition 3.1. *[9] A stationary framework* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbb{P})$ *is said to be ergodic if*

$$\lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathbb{P}(A_1 \cap \theta_{-x} A_2) dx = \mathbb{P}(A_1)\mathbb{P}(A_2),\ \forall A_1, A_2, \in \mathcal{A}.$$

It is said to be mixing if

$$\lim_{|x|\to\infty} \mathbb{P}(A_1 \cap \theta_{-x} A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2),\ \forall A_1, A_2 \in \mathcal{A}.$$

We give the following Lemma without a proof.

Lemma 3.1. *[9] If a stationary framework* $(\Omega, \mathcal{A}, \{\theta_t\}_{t\in\mathbb{R}^d}, \mathbb{P})$ *is mixing, then it is ergodic.*

Definition 3.2. *[9] A stationary random measure* Φ *on* $\mathbb{R}^d$ *is said to be ergodic (respectively mixing) if and only if the stationary framework* $(\mathbb{M}(\mathbb{R}^d), \mathcal{M}(\mathbb{R}^d), \{S_t\}_{t\in\mathbb{R}^d}, \mathbb{P}_\Phi)$ *is ergodic (respectively mixing).*

Proposition 3.1. *[9] Let* Φ *be a stationary random measure with Laplace transform* $\mathcal{L}_\Phi$*. Then*

(i) Φ *is ergodic if and only if*

$$\lim_{a\to\infty} \frac{1}{(2a)^d} \int_{[-a,a]^d} \mathcal{L}_\Phi(f_1 + S_x f_2) dx = \mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(f_2).$$

(ii) Φ *is mixing if and only if*

$$\lim_{|x|\to\infty} \mathcal{L}_\Phi(f_1 + S_x f_2) = \mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(f_2),$$

for all measurable $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}_+$ *bounded with bounded support.*

Corollary 3.1. *[9] A homogeneous Poisson point process on* $\mathbb{R}^d$ *is mixing and ergodic.*

Proof. Let Φ be a homogeneous Poisson point process on $\mathbb{R}^d$ and let $f_1, f_2 : \mathbb{R}^d \to \mathbb{R}_+$ be measurable bounded with bounded support. Let $B_1, B_2 \in \mathcal{B}_c(\mathbb{R}^d)$ be their respective supports. For $|x|$ sufficiently large, the sets B_1 and $B_2 + x$ are disjoint, so the restrictions of Φ to these sets are independent. For such x

$$\begin{aligned}\mathcal{L}_\Phi(f_1 + S_x f_2) &= \mathbb{E}\Big[\exp\Big(-\int_{\mathbb{R}^d}(f_1 + S_x f_2)d\Phi\Big)\Big] \\ &= \mathbb{E}\Big[\exp\Big(-\int_{B_1} f_1 d\Phi\Big)\exp\Big(-\int_{B_2} S_x f_2 d\Phi\Big)\Big] \\ &= \mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(S_x f_2) = \mathcal{L}_\Phi(f_1)\mathcal{L}_\Phi(f_2),\end{aligned} \tag{3.48}$$

where the last equality holds because of stationarity. Thus Φ is mixing by the Proposition 3.1(ii). From Lemma 3.1 it follows that it is ergodic. □

Lemma 3.2. *The follower point process of order* n *is a factor of a Poisson point process, and hence is mixing.*

Discrete event spatial simulation starts with sampling a Poisson point process in the fixed bounded window $W \subset \mathbb{R}^2$ (the observation window). We pick a convex W, typically a square. In order to simulate a homogeneous Poisson point process, we use the ideas from [5]. In [5] it is suggested that simulating in a compact region W can be split into two parts. Namely, the number of points in W is determined from a simulation of a Poisson random variable. Then the positions of points in W are obtained from simulating a binomial point process with that number of points. Since we are interested in the point process in the whole of $\mathbb{R}^2$, we take W big, hence there are small boundary effects.

Since the homogeneous Poisson point process is scale invariant and ergodic, on average what we see in the large observation window W that allows one to evaluate the desired probabilities. Then we apply the Follower Dynamics to all the points simultaneously. We still have mixing properties satisfied. All simulations are done using MATLAB. For each agent, we keep track of its leader at every step. In order to calculate any frequency, we count the number of agents for which our sets of conditions are satisfied and we divide by the total number of agents. In order to obtain a 95% confidence interval, we run the simulation many times (insert how many and over how many points).

For example for the frequency of ultimate leader pairs of order 0, we check for all pairs of agents and count the number of mutually closest neighbors at step 0. Since we keep track of the leader for each agent, we just check whether two agents are each others' leaders.

Similarly, in order to determine the frequency of the ultimate leader pairs of order 1 type 1, we check for all pairs of agents, whether they had the same leader (but are not each others' leaders) at step 0 and are each others' leaders at step 1.

For the frequency of the ultimate leader pairs of order 1 type 2, we check, for all pairs of agents, whether one was a leader of the other at step 0 and both are each others' leaders at step 1.

Table 3.3.1 contains frequencies and 95% confidence interval obtained using discrete event spatial simulation. In order to obtain a 95% confidence interval we run simulation many times and keep track of the statistics. The statistics is obtained from 40 samples containing in mean each 20000 points. We observe that the frequency of ultimate leader pairs of order 1 is $[0.021, 0.023]$. Similarly, frequency of inversion is $[0.13, 0.14]$.

Frequency of	95% Confidence interval
Ultimate leader pair of order 0	$[0.6203, 0.6227]$
Order 1 type 1	$[0.0113, 0.0120]$
Order 1 type 2	$[0.0100, 0.0105]$
4 body swap	$[0.00006334, 0.0001]$
Inversion Type 1	$[0.1381, 0.1392]$
Inversion Type 2	$[0.0001899, 0.0002687]$

3.3.2 Integral Equations and Semi-Algebraic Sets

When exploiting our integral geometry representations of frequencies, we end up with expressions that are integrals over unions of balls. We hence need to compute the volumes of the union of balls. First, we give some more background. In this subsection we first represent the unions of balls as a semi-algebraic set.

The setting is $\mathbb{R}^d$. Let k be a positive integer. Let $c_1, \ldots, c_k$ be arbitrary points of $\mathbb{R}^d$ which are the centers of balls of positive radii $r_1, \ldots, r_k$, respectively. We are interested in the volume of

$$\mathbb{U} := \cup_{i=1}^{k} B(c_i, r_i),$$

with $B(c, r)$ the closed ball of center c and radius r.

Let $\mathcal{S}$ be set of all non-empty subsets of $[1, \ldots, k]$ For all $s \in S$, let

$$\mathbb{V}_s = \cap_{i \in s}^{k} B(c_i, r_i) \cap_{j \notin s}^{k} B(c_j, r_j)^c.$$

We have

$$\mathbb{U} = \cup_{s\in S}\mathbb{V}_s,$$

where the last union is disjoint. Hence

$$\mid \mathbb{U} \mid = \sum_{s\in S} \mid \mathbb{V}_s \mid .$$

But for all s, $\mathbb{V}_s$ is the following semi-algebraic set:

$$\begin{aligned} ||z - x_i|| &\leq r_i, \quad i \in s \\ ||z - x_j|| &> r_j, \quad i \notin s. \end{aligned}$$

Hence $\mathbb{U}$ is the disjoint union of the $2^k - 1$ semi-algebraic sets $\{\mathbb{V}_s, s \in S\}$ and its volume is the sum of the volumes of these semi-algebraic sets of $\mathbb{R}^d$.

Once we have a semi-algebraic set we need to compute an integral over it.

We apply this analysis to the each case mentioned in above section and give the numerical values.

Ultimate Leader Pair of Order 1, Type 1 In order to evaluate the frequency of the ultimate leader pair of order 1, type 1, we need to get to the integral over the semi-algebraic set. We start with the integral for β^1 derived in Sections 3.1.2:

$$\begin{aligned} \mathbb{E}[N_1^{(4)}(\mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 1_{\tilde{x}_1,\tilde{x}_2,\tilde{x}_4 \in \tilde{\mathcal{D}}} \\ e^{-\lambda(\mathrm{Vol}(B(0\to\tilde{x}_4)\cup B(\tilde{x}_1\to 0)\cup B(\tilde{x}_2\to 0))}\lambda^4 d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4 = |C|\beta^1, \end{aligned}$$

with $\tilde{\mathcal{D}} = \{(x_1 - x_3, x_2 - x_3, 0, x_4 - x_3) : (x_1, x_2, x_3, x_4) \in \mathcal{D}\}$.

In order to evaluate the integral, we need to write a formula for the volume of the union of 3 balls. An illustration of the 3 balls is given in Figure 3.4.

Using the inclusion exclusion principle we can express the volume $\mathbb{U}$ of the union as follows:

$$\begin{aligned} \mathbb{U} = \mathrm{Vol}(B(0 \to \tilde{x}_4)) + \mathrm{Vol}(B(\tilde{x}_1 \to 0)) + \mathrm{Vol}(B(\tilde{x}_2 \to 0)) - \\ \mathrm{Vol}(B(0 \to \tilde{x}_4) \cap B(\tilde{x}_1 \to 0)) - \mathrm{Vol}(B(0 \to \tilde{x}_4) \cap \mathrm{Vol}(B(\tilde{x}_2 \to 0)) - \\ \mathrm{Vol}(B(\tilde{x}_1 \to 0) \cap (B(\tilde{x}_2 \to 0)) + \mathrm{Vol}(B(0 \to \tilde{x}_4) \cap B(\tilde{x}_1 \to 0) \cap (B(\tilde{x}_2 \to 0)). \end{aligned} \tag{3.49}$$

We see that, as announced, the frequency β^1 of phenomenon M_1 is obtained as an integral over a semi-algebraic set.

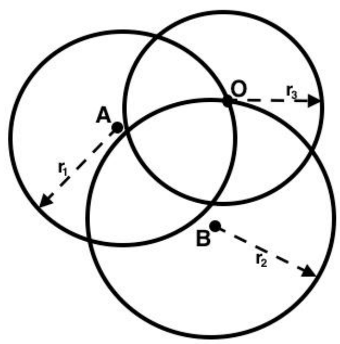


Figure 3.4: 3 balls with radii r_1, r_2, and r_3.

More precisely, in order to calculate the frequency, we need expressions for the volume of intersection of two balls and the volume of intersection of three balls. Note that if we know the coordinates $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_4$, we can calculate the radii of the circles as

$$r_1 = d(\tilde{x}_1, 0),\ r_2 = d(\tilde{x}_2, 0),\ r_3 = d(0, \tilde{x}_4).$$

The distance between agents A ($\tilde{x}_1$) and B ($\tilde{x}_2$) is denoted by $d_c = d(\tilde{x}_1, \tilde{x}_2)$.

Let us denote by $V_{int1} = \mathrm{Vol}(B(0 \to \tilde{x}_4) \cap B(\tilde{x}_1 \to 0))$, the volume of the intersection of the two balls, one centered at $\tilde{x}_1$ of radius r_1, and the other at 0 of radius r_3.

$$V_{int1} = r_3^2 \cos^{-1}\left(\frac{r_3}{2r_1}\right) + r_1^2 \cos^{-1}\left(\frac{2r_1^2 - r_3^2}{2r_1^2}\right) - \frac{1}{2}\sqrt{r_3^2((2r_1)^2 - r_3^2)}.$$

Similarly, denote by $V_{int2} = \mathrm{Vol}(B(0 \to \tilde{x}_4) \cap \mathrm{Vol}(B(\tilde{x}_2 \to 0))$, the volume of the intersection of the two balls one centered at $\tilde{x}_2$ of radius r_2 and the other at 0 of radius r_3.

$$V_{int2} = r_3^2 \cos^{-1}\left(\frac{r_3}{2r_2}\right) + r_2^2 \cos^{-1}\left(\frac{2r_2^2 - r_3^2}{2r_1^2}\right) - \frac{1}{2}\sqrt{r_3^2((2r_2)^2 - r_3^2)}.$$

Finally, denote by $V_{int3} = \mathrm{Vol}(B(\tilde{x}_1 \to 0) \cap (B(\tilde{x}_2 \to 0))$ the volume of the intersection of the two balls, one centered at $\tilde{x}_1$ of radius r_1, and the other at $\tilde{x}_2$ of radius r_2.

$$\begin{aligned} V_{int3} = r_1^2 \cos^{-1}\left(\frac{d_c^2 + r_1^2 - r_2^2}{2d_c r_1}\right) + r_2^2 \cos^{-1}\left(\frac{d_c^2 + r_2^2 - r_1^2}{2d_c r_2}\right) - \\ \frac{1}{2}\sqrt{(-d_c + r_1 + r_2)(d_c + r_1 - r_2)(d_c - r_1 + r_2)(d_c + r_1 + r_2)}. \end{aligned} \tag{3.50}$$

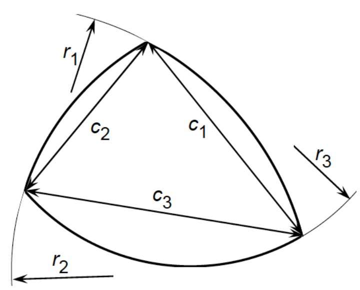


Figure 3.5: Image from [53]. Geometry of the intersection of 3 circles, a circular triangle. The radii and the chords involved are shown.

We just need an expression for the volume of the intersection of three balls. We use analysis from the paper [53] to get the formula. An illustration from

[53] is shown in Figure 3.5. Denoting the intersection $V_{allint} = \mathrm{Vol}(B(0 \to \tilde{x}_4) \cap B(\tilde{x}_1 \to 0) \cap (B(\tilde{x}_2 \to 0))$. A general expression that includes the radii and the chords from [53](using the notation of Figure 3.5):

$$
\begin{aligned}
V_{allint} = \frac{1}{4}\sqrt{(c_1+c_2+c_3)(c_2+c_3-c_1)(c_1+c_2-c_3)(c_1-c_2+c_3)} \\
+\sum_{k=1}^{3}\left(r_k^2 \sin^{-1}\frac{c_k}{2r_k} - \frac{c_k}{4}\sqrt{4r_k^2-c_k^2}\right).
\end{aligned}
\tag{3.51}
$$

It takes some work to calculate the exact expressions for the chords since we need to find the points of intersection of the circles. Several cases are illustrated in Figure 3.6. It is clear that not all configurations that we're interested in, would even have a circular triangle intersection. However, with careful analysis like in [53], given the coordinates of the 3 agents, there is an algorithm to calculate the area of intersection of 3 balls.

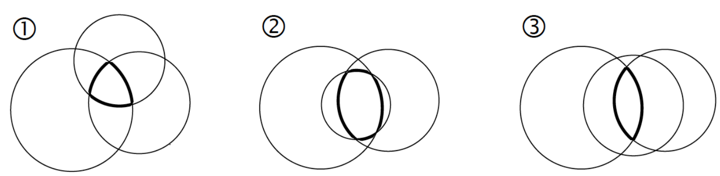


Figure 3.6: Image from [53]. Different cases when 3 circles intersect. In case ② the intersection produces a circular quadrilateral. In case ③ it is basically the intersection of the two outer circles.

Now that we have all the elements, we have a formula to calculate exact value of the volume of the union of balls $\mathbb{U}$. For compactness, the integral of the frequency of ultimate leader pairs of order 1, type 1 can be written as:

$$
\begin{aligned}
\mathbb{E}[N_1^{(4)}(\mathbb{R}^2 \times \mathbb{R}^2 \times C \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \lambda^4 1_{\tilde{x}_1,\tilde{x}_2,\tilde{x}_4 \in \tilde{\mathcal{D}}} \\
e^{-\lambda\mathbb{U}} d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_4,
\end{aligned}
$$

with $\tilde{\mathcal{D}} = \{(x_1 - x_3,\, x_2 - x_3,\, 0,\, x_4 - x_3) : (x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}\}$.

Numerical Result: In practice the process of calculating these integrals is deterministic. However, the algorithm is quite slow $O(N^6)$, where N is the number of spacing. The algorithm for calculating the frequency of the upper bound of the ultimate leader pairs of order 1, type 1 was written in Python and a value of around 0.03 was obtained. This is comparable and slightly larger than the simulation results. Recall that the simulation counts that the frequency for the ultimate leader pairs of order 1, type 1 and the integral is an upper bound calculation. Even this upper bound estimate takes several hours to run in order to obtain the approximate value of the integral. Because of the speed of the algorithm and the time constraints we do not have confidence intervals for this calculation. The algorithm can be found in the Appendix, Section B for this chapter.

Ultimate Leader Pair of Order 1, Type 2 Similarly for the ultimate leader pair of order 1, type 2 we have the following analysis.

$$\mathbb{E}[N_1^{(4)}(\mathbb{R}^2 \times C \times \mathbb{R}^2 \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} 1_{\tilde{x}_1, \tilde{x}_2, \tilde{x}_4 \in \tilde{\mathcal{D}}}$$
$$e^{-\lambda(\mathrm{Vol}(B(\tilde{x}_3 \to \tilde{x}_4) \cup B(\tilde{x}_1 \to 0) \cup B(0 \to \tilde{x}_3))} \lambda^4 d\tilde{x}_1 d\tilde{x}_3 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}} = \{(x_1 - x_2,\, 0,\, x_3 - x_2,\, x_4 - x_2) : (x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}\}$.

In order to evaluate the integral, we need to write a formula for the volume of the union of 3 balls. Analysis is the same as the one for the type 1 case. Figure 3.7 illustrates two possible cases.

Using the inclusion exclusion principle, we express the volume $\mathbb{U}$ of the

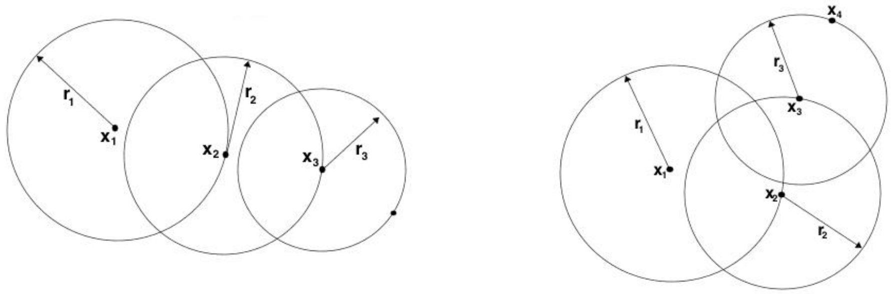


Figure 3.7: Illustration of two different cases of union of 3 balls for type 2.

union as follows:

$$\begin{aligned}\mathbb{U} = \mathrm{Vol}(B(\tilde{x}_3 \to \tilde{x}_4)) + \mathrm{Vol}(B(\tilde{x}_1 \to 0)) + \mathrm{Vol}(B(0 \to \tilde{x}_3)) - \\ \mathrm{Vol}(B(0 \to \tilde{x}_3) \cap B(\tilde{x}_1 \to 0)) - \mathrm{Vol}(B(\tilde{x}_3 \to \tilde{x}_4) \cap \mathrm{Vol}(B(\tilde{x}_1 \to 0)) - \\ \mathrm{Vol}(B(\tilde{x}_3 \to \tilde{x}_4) \cap (B(0 \to \tilde{x}_3)) + \mathrm{Vol}(B(0 \to \tilde{x}_3) \cap B(\tilde{x}_1 \to 0) \cap (B(\tilde{x}_3 \to \tilde{x}_4)).\end{aligned} \tag{3.52}$$

We see that as announced the upper bound frequency, β^2, of phenomenon M_2 is obtained as an integral over a semi-algebraic set.

$$r_1 = d(\tilde{x}_1, 0),\ r_2 = d(0, \tilde{x}_3),\ r_3 = d(\tilde{x}_3, \tilde{x}_4).$$

Distance between the agents $\tilde{x}_1$ and $\tilde{x}_3$ is denoted as $d_c = d(\tilde{x}_1, \tilde{x}_3)$.

Let us denote by $V_{1int} = \mathrm{Vol}(B(0 \to \tilde{x}_3) \cap B(\tilde{x}_1 \to 0))$ the volume of the intersection of the two balls one centered at $\tilde{x}_1$ of radius r_1 and the other at 0 of radius r_2.

$$V_{int1} = r_2^2 \cos^{-1}\left(\frac{r_2}{2r_1}\right) + r_1^2 \cos^{-1}\left(\frac{2r_1^2 - r_2^2}{2r_1^2}\right) - \frac{1}{2}\sqrt{r_2^2((2r_1)^2 - r_2^2)}.$$

Similarly, denote by $V_{2int} = \mathrm{Vol}(B(\tilde{x}_3 \to \tilde{x}_4) \cap \mathrm{Vol}(B(\tilde{x}_1 \to 0))$ the volume of the intersection of the two balls one centered at $\tilde{x}_3$ of radius r_3 and the other at $\tilde{x}_1$ of radius r_1.

$$\begin{aligned} V_{2int} &= r_3^2 \cos^{-1}\left(\frac{d_c^2 + r_3^2 - r_1^2}{2d_c r_3}\right) + r_1^2 \cos^{-1}\left(\frac{d_c^2 + r_1^2 - r_3^2}{2d_c r_1}\right) - \\ &\frac{1}{2}\sqrt{(-d_c + r_3 + r_1)(d_c + r_3 - r_1)(d_c - r_3 + r_1)(d_c + r_3 + r_1)}. \end{aligned} \tag{3.53}$$

Finally, denote by $V_{3int} = \mathrm{Vol}(B(\tilde{x}_3 \to \tilde{x}_4) \cap (B(0 \to \tilde{x}_3))$ the volume of the intersection of the two balls one centered at $\tilde{x}_3$ of radius r_3 and the other at 0 of radius r_2.

$$V_{3int} = r_2^2 \cos^{-1}\left(\frac{r_2}{2r_3}\right) + r_3^2 \cos^{-1}\left(\frac{2r_3^2 - r_2^2}{2r_3^2}\right) - \frac{1}{2}\sqrt{r_2^2((2r_3)^2 - r_2^2)}.$$

We just need an expression for the volume of intersection of three circles. Like shown in left Figure 3.7, in some configurations that intersection is empty. Again, we refer to the analysis from the paper [53] to get the formula. An illustration from [53] is shown in Figure 3.5. Denoting the intersection $V_{allint} = \mathrm{Vol}(B(0 \to \tilde{x}_3) \cap B(\tilde{x}_1 \to 0) \cap (B(\tilde{x}_3 \to \tilde{x}_4))$. Here is a general expression that includes the radii and the chords from [53](using the notation of Figure 3.5):

$$\begin{aligned} V_{allint} = \frac{1}{4}&\sqrt{(c_1 + c_2 + c_3)(c_2 + c_3 - c_1)(c_1 + c_2 - c_3)(c_1 - c_2 + c_3)} \\ &+ \sum_{k=1}^{3}\left(r_k^2 \sin^{-1}\frac{c_k}{2r_k} - \frac{c_k}{4}\sqrt{4r_k^2 - c_k^2}\right). \end{aligned} \tag{3.54}$$

Combining together all the volumes written above, we have a formula for the exact value of the volume of the union of balls $\mathbb{U}$. The integral of the frequency of ultimate leader pairs of order 1, type 2 can be written as:

$$\mathbb{E}[N_1^{(4)}(\mathbb{R}^2 \times C \times \mathbb{R}^2 \times \mathbb{R}^2)] = |C| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \lambda^4 1_{\tilde{x}_1, \tilde{x}_2, \tilde{x}_4 \in \tilde{\mathcal{D}}} e^{-\lambda \mathbb{U}} d\tilde{x}_1 d\tilde{x}_3 d\tilde{x}_4,$$

with $\tilde{\mathcal{D}} = \{(x_1 - x_2,\, 0,\, x_3 - x_2,\, x_4 - x_2) : (x_1,\, x_2,\, x_3,\, x_4) \in \mathcal{D}\}$.

Integral geometric calculations for other cases are similar.

3.4 Generalizations

In the same manner, in principle, we can write a set of conditions in order to calculate the intensity of ultimate leader pairs of order n, i.e., the probability that two points were not ultimate leader pairs in the first $n-1$ iterations but are at the n^{th} step. However it is hard to actually calculate this integral and that is why we use numerical methods in order to estimate the intensities of party swaps of higher order. Certain upper bounds are also used to estimate relevant intensities since they are easier to handle.

Chapter 4

Lack of Percolation

Alice: This is impossible.
The Mad Hatter: Only if you believe it is.

Lewis Carroll, Alice in Wonderland

In this chapter, we analyze the a.s. finiteness of the follower parties at different steps.

The chapter is divided into 2 sections. In Section 4.1, we give the necessary notation and a short overview of the percolation concepts used in this chapter. Then, in Section 4.2 we give a novel proof that uses percolation theory and the mass transport formula to prove that the Poisson descending tree is finite a.s.. We leverage the fact that each infinite path in a descending tree is an infinite sequence $x_1, x_2, ...$ of mutually different points of Φ for which $|x_{j-1} - x_j| \geq |x_j - x_{j+1}|$, for all $j \geq 2$. We show how this proof method can be extended to prove a.s. finiteness of follower parties at step 1, and possibly further steps.

4.1 Definitions and Notation

We list additional notation used throughout this chapter. Let $h(x, \Phi)$ denote the direct leader of x in Φ. Similarly $h^2(x, \Phi)$ denotes the direct leader of the leader of x in Φ, etc.

Let us start with the needed background on percolation theory. General references on lattice percolation can be found in [34, 35]. We will tessellate the space with the square grid and will use the discrete percolation theory for our proof. Therefore we need to give some background first.

Let V denote the vertex set of $\mathbb{Z}^2$. A *site percolation measure* on an infinite graph G is a probability measure on the space of assignments of a state, namely open or closed, to each vertex of G (here $\Omega = \prod_{v \in V}\{0, 1\}$, $\mathcal{F}$ is the σ-algebra generated by the cylinder sets, i.e., the events that only depend on a finite number of vertices). We will focus particularly on the square lattice $\mathbb{Z}^2$ (Figure 4.1).

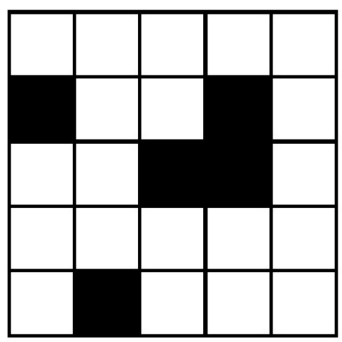


Figure 4.1: Site percolation example, open squares are colored white and closed are colored black.

An *open cluster* is a maximal connected subgraph of $\mathbb{Z}^2$ all of whose vertices are open. We write C_v for the open cluster containing a given vertex $v \in \mathbb{Z}^2$. Thus a vertex w lies in C_v if and only if w can be reached from v by an open path, i.e., a path in $\mathbb{Z}^2$ all of whose vertices are open. In the case of site percolation, if v is closed then $C_v = \emptyset$. Writing $|C_v|$ for the number of vertices of C_v, let

$$\theta(p) = \mathbb{P}_p(|C_0| = \infty),$$

where $0 = (0, 0)$ is the origin. By Kolmogorov's 0-1 law, percolation occurs if and only if $\theta(p) > 0$. More precisely, if $\theta(p) > 0$, then with probability 1, there is an infinite open cluster somewhere in $\mathbb{Z}^2$, while if $\theta(p) = 0$, then, with probability 1, there is no such cluster. The critical probability depends on the lattice and the type of percolation that is being looked at. We are interested in dependent percolation on the square lattice $a\mathbb{Z}^2$, $a > 0$.

Definition 4.1. *A bond percolation measure on a graph G is k-dependent if, for every pair S, T of sets of edges of G at graph distance at least k, the states (being open or closed) of the edges in S are independent of the states of the edges in T . When $k = 1$, the separation condition is exactly that no edge of S shares a vertex with an edge of T. The definition of k-dependence for a site percolation measure on G is exactly the same, except that S and T run over all sets of vertices at graph distance at least k. Again, we will only consider dependent measures on the lattice $a\mathbb{Z}^2$.*

4.2 The Follower Model and Percolation

This section contains our main results and proofs. We first give a novel proof that the Poisson Descending Chain is finite a.s. We then extend the result to the Poisson Follower Model graph at step 1.

We give an overview of the ideas developed for this problem. They will apply for each step of the Follower Dynamics with some technical differences for each step.

We tessellate the plane with squares of side length $2a$ and each point of the

Follower Model is in some square. Based on certain local conditions, we declare squares open or closed. We can show that the open squares percolate and the closed ones do not. Therefore, if there is an infinite party that contains the origin, then looking at the forward set from 0, it has to cross an open square at some point. This proof fundamentally leverages that the distances are decreasing, which is why we look at the forward set from 0. Once it hits one open square, the forward set cannot "travel" too far. Namely it can only cross finitely many open squares and closed squares. We get that the forward set has to be finite a.s. Then using the Point Map Classification [29] we conclude that the party is finite a.s.

4.2.1 Follower Parties at Step 0

Theorem 4.1. *The nearest neighbor graph on a Poisson point process does not percolate.*

Proof. The proof consists in first showing that the forward set of a typical point is integrable a.s. Then, using the Mass Transport Principle [9], we prove that the backward (or the follower set) is also finite a.s.. The proof relies on the property of the Follower Model that every agent always follows its nearest neighbor, so the distances are always decreasing along the forward path.

Let $\epsilon > 0$ be fixed.

For a point $x \in \Phi$ define its *contraction ratio* $\rho(x)$ as

$$\rho(x) = \frac{d(h(x,\Phi), h^2(x,\Phi))}{d(x, h(x,\Phi))} 1_{h^2(x,\Phi) \neq x}. \tag{4.1}$$

We exclude the nearest neighbor pairs from consideration for the ratio. For mutually closest neighbors, we define the contraction ratio to be 0.

Theorem 4.2. *Under $\mathbb{P}^0$ the forward and the backward sets of* 0 *are* a.s. *finite.*

Proof. The proof is based on a series of lemmas. Before we begin the proof: In this theorem, a path that ends with a mutually closest neighbor pair is considered infinite.

The forward path can end at finite distance with a mutually closest neighbor pair and the statement would be true, or it doesn't end that way. If it does not, let us consider the following site percolation problem:

Define the square boxes to be of length $2a$ and center $2na$, $n \in \mathbb{Z}^2$. Declare a square *open* if it satisfies the following conditions:

i It contains at least one point;

ii Each point in the square has its nearest neighbor at a distance at most a, where $2a$ is the side length of the squares;

iii Each point in the square has a contraction ratio of at most ρ.

A square is *closed* if it is not open. In Lemma 4.1 we give an explicit calculation of the probability that a point has a contraction ratio greater than $1-\epsilon$, where ϵ is a constant we can vary. Then in Lemma 4.2, we show that the probability that a square is closed can be made arbitrarily close to 0 by choosing a large enough and ϵ small enough.

The structure of the proof of Theorem 4.2 is divided into three separate components:

1. For a proper choice of lattice side length $2a$, there is a percolation of the open squares;

2. Each infinite path from 0 crosses infinitely many open squares;

3. Use mass transport formula to show that the forward path from 0 has to be finite and then show that the backward set from 0 also has to be finite a.s.

Lemma 4.1. *For all a, the Palm probability that the distance to the nearest point is less than a and that the contraction ratio of the origin is greater than $1-\epsilon$ tends to 0 as $\epsilon \to 0$.*

Proof. We use the notation represented in Figure 4.2. Denote the distance from x to its nearest neighbor y, by r_1. Let the distance $d(h(x,\Phi), h^2(x,\Phi))$ be r_2. For all $r_2 < r_1$, let A_1 be the region $B(y,r_2) \setminus B(x,r_1)$, and A_2 be the region $B(y,r_1) \setminus B(y,r_2)$. For the contraction ratio to be more than $1-\epsilon$, we need region A_1 to be empty of other points, and A_2 to contain at least one point for $r_2 = r_1(1-\epsilon)$.

Denote by p_0 the probability of the event that point x has a nearest neighbor at distance r_1 and the neighbor of x has its nearest neighbor at a distance between r_2 and r_1.

We are only interested in the case when $r_1 < a$, so that the probability $p_0(a)$ can be written as follows:

$$p_0(a,\epsilon) = \int_0^a \pi[\text{nearest neighbor is at distance } r_1 < a]$$
$$\mathbb{P}[A_1 \text{ is empty of other points}]\mathbb{P}[A_2 \text{ contains at least 1 point}]dr_1,$$

where π is the Poisson nearest neighbor density, i.e., the Rayleigh density.

In order to get explicit probabilities, we first need to calculate different areas. Let us denote by $A_{intersection}$, the area of the intersection of two balls,

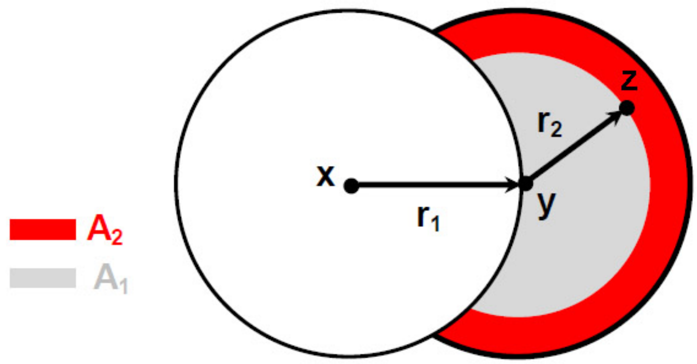


Figure 4.2: Contraction ratio $\rho(x) = r_2/r_1$. Gray shaded area depicts A_1 and red shaded area depicts A_2

one centered at x of radius r_1, and the other at y of radius r_2.

$$A_{intersection} = r_2^2 \cos^{-1}\left(\frac{r_2}{2r_1}\right) + r_1^2 \cos^{-1}\left(\frac{2r_1^2 - r_2^2}{2r_1^2}\right) - \frac{1}{2}\sqrt{r_2^2((2r_1)^2 - r_2^2)}.$$

If $r_1 = r_2$ then the area of intersection simplifies to $A_{eq} = r_1^2(\frac{2\pi}{3} - \frac{\sqrt{3}}{2})$.

The area of region A_1 is

$$A_1 = r_2^2\pi - A_{intersection}.$$

Similarly, the area of region A_2 is

$$A_2 = (r_1^2\pi - A_{eq}) - A_1 = (r_1^2\pi - A_{eq}) - r_2^2\pi + A_{intersection}.$$

The total probability of a point having contraction ratio greater than $1-\epsilon$ is hence

$$p_0(a,\epsilon) = \int_0^a 2\lambda\pi r_1 e^{-\lambda r_1^2\pi} \cdot e^{-\lambda A_1} \cdot (1 - e^{-\lambda A_2}) dr_1,$$

with $r_2 = r_1(1-\epsilon)$. Since $\epsilon \ll 1$, throughout the calculations below we apply a first order Taylor expansions; with $(1-\epsilon)^2 \approx 1 - 2\epsilon$ and $\sqrt{1+\epsilon} \approx 1 + \frac{1}{2}\epsilon$.

Combining the equations for A_1, A_2 and A_{eq} all together into p_0 we write it as a difference of two integrals

$$p_0(a,\epsilon) = \int_0^a 2\lambda\pi r_1 e^{-\lambda r_1^2\pi}\cdot e^{-\lambda A_1}dr_1 - \int_0^a 2\lambda\pi r_1 e^{-\lambda r_1^2\pi}\cdot e^{-\lambda(A_1+A_2)}dr_1.$$

Let $I_1 = \int_0^a 2\lambda\pi r_1 e^{-\lambda r_1^2\pi}\cdot e^{-\lambda A_1}dr_1$ and $I_2 = \int_0^a 2\lambda\pi r_1 e^{-\lambda r_1^2\pi}\cdot e^{-\lambda(A_1+A_2)}dr_1$.

In calculations that follow, we again use $\lambda = 1$.

We calculate each integral separately and then combine them together to find

$$p_0(a,\epsilon) = I_1 - I_2.$$

First we can simplify the expression for A_1:

$$A_1 = r_1^2\Big((1-\epsilon)^2(\cos^{-1}(\frac{1-\epsilon}{2})-\pi)+\cos^{-1}(1-\frac{(1-\epsilon)^2}{2})-\frac{1-\epsilon}{2}\sqrt{4-(1-\epsilon)^2}\Big).$$

Using a Taylor series expansion of first order, we can approximate A_1 as

$$A_1 \approx r_1^2\Big((1-2\epsilon+o(\epsilon))(-\frac{2\pi}{3}+\frac{\epsilon}{\sqrt{3}}+o(\epsilon))+\frac{\pi}{3}-\frac{9\epsilon}{8}-\frac{\sqrt{3}}{2}(1-\frac{2\epsilon}{3}+o(\epsilon))\Big),$$

which simplifies to

$$A_1 \approx r_1^2\Big(-\frac{\pi}{3}-\frac{\sqrt{3}}{2}+\epsilon(\frac{4\pi}{3}-\frac{9}{8})+o(\epsilon)\Big).$$

Thus I_1 is approximately

$$\begin{aligned} I_1 &\approx \int_0^a 2\pi r_1 e^{-r_1^2\pi} e^{r_1^2(-\frac{\pi}{3}-\frac{\sqrt{3}}{2}+\epsilon(\frac{4\pi}{3}-\frac{9}{8})+o(\epsilon))}dr_1. \\ &\approx \int_0^a 2\pi r_1 e^{-r_1^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}-\epsilon(\frac{4\pi}{3}+\frac{9}{8})+o(\epsilon))}dr_1. \end{aligned} \tag{4.2}$$

Using the substitution $u = r_1^2(\frac{4\pi}{3} + \frac{\sqrt{3}}{2} - \epsilon(\frac{4\pi}{3} + \frac{9}{8}) + o(\epsilon))$, we get that

$$I_1 \approx \frac{\pi}{(\frac{4\pi}{3} + \frac{\sqrt{3}}{2} - \epsilon(\frac{4\pi}{3} + \frac{9}{8}) + o(\epsilon))}\Big(1 - e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}-\epsilon(\frac{4\pi}{3}+\frac{9}{8})+o(\epsilon))}\Big).$$

With another application of a Taylor expansion, we can write I_1 in terms of ϵ as

$$\begin{aligned} I_1 \approx \frac{\pi}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}}\Big(1 - e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}-\epsilon(\frac{4\pi}{3}+\frac{9}{8})+o(\epsilon))}\Big)+ \\ \epsilon\frac{\pi}{(\frac{4\pi}{3} + \frac{\sqrt{3}}{2})^2}\Big(1 - e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}-\epsilon(\frac{4\pi}{3}+\frac{9}{8})+o(\epsilon))}\Big) + o(\epsilon). \end{aligned} \tag{4.3}$$

Plugging the explicit formulas for A_1 and A_2, we get that I_2 simplifies to

$$I_2 = \int_0^a 2\pi r_1 e^{-r_1^2\pi} e^{-r_1^2(\frac{\pi}{3}+\frac{\sqrt{3}}{2})} dr_1.$$

Applying the substitution $u = r_1^2(\frac{4\pi}{3} + \frac{\sqrt{3}}{2})$, we end up with

$$I_2 = \frac{\pi}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}}\Big(1 - e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})}\Big).$$

Combining the calculations for I_1 and I_2, we have that the probability p_o is approximately

$$\begin{aligned} p_0(a,\epsilon) \approx \frac{\pi}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}} \cdot e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})} \cdot \Big(1 - e^{-a^2(\epsilon(\frac{4\pi}{3}+\frac{9}{8})+o(\epsilon))}\Big) \\ +\epsilon\frac{\pi}{(\frac{4\pi}{3} + \frac{\sqrt{3}}{2})^2}\Big(1 - e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}-\epsilon(\frac{4\pi}{3}+\frac{9}{8})+o(\epsilon))}\Big) + o(\epsilon). \end{aligned} \tag{4.4}$$

Applying the Taylor expansion to the exponential functions, we get

$$\begin{aligned} p_0(a,\epsilon) \approx \frac{\pi}{\frac{4\pi}{3} + \frac{\sqrt{3}}{2}} e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})}\Big(a^2(\epsilon(\frac{4\pi}{3} + \frac{9}{8}) + o(\epsilon)\Big)+ \\ \epsilon\frac{\pi}{(\frac{4\pi}{3} + \frac{\sqrt{3}}{2})^2}\Big(1 - e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2}-\epsilon(\frac{4\pi}{3}+\frac{9}{8})+o(\epsilon))}\Big) + o(\epsilon) \end{aligned} \tag{4.5}$$

Simplifying even more,

$$p_0(a,\epsilon) \approx \epsilon \cdot \frac{a^2\pi}{\frac{4\pi}{3}+\frac{\sqrt{3}}{2}}\left(e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})}+\frac{4\pi}{3}+\frac{17}{8}\right)+o(\epsilon),$$

where the second part of the product is a constant that depends on a. This implies that for each a, there exists ϵ small enough, which makes p_0 as small as needed.

□

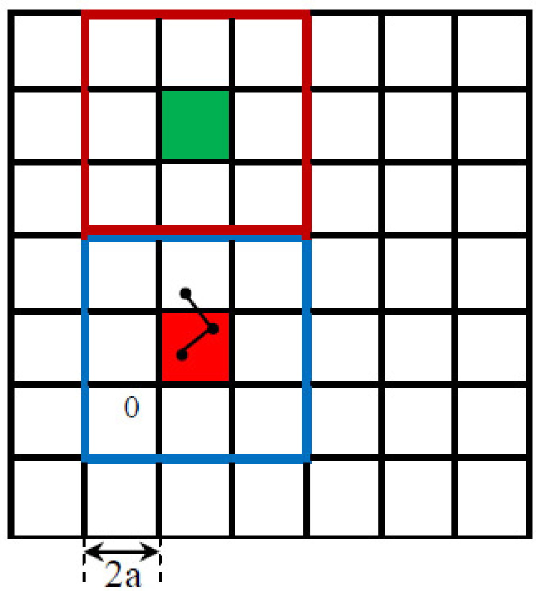


Figure 4.3: The red square is closed if it or one of its 8 neighbors contains a point with a contraction ratio greater than $1-\epsilon$. Since the distance to the nearest neighbor is at most a, and the square has side length $2a$, starting from the red square we cannot cross the blue edge in 2 steps. The states of the red and the green square are independent of each other. By construction, the states of the squares and their "shield" area need to be disjoint. Thus we have a 3-dependent setting.

Component 1: Take a point x from one of the open squares of side length $2a$. The state of a square depends on the points in the neighboring squares. Let us denote neighboring squares as layer 1 and the square in question, layer 0. Since we want layers 0 and 1 to be independent of other similar squares, we have a 3-dependent site percolation measure. This is illustrated in Figure 4.3, where the state of the red square is independent of the state of the green square.

In Lemma 4.1 we calculated the probability that the distance to the nearest point is less than a and that the contraction ratio of the origin is greater than $1 - \epsilon$. In Lemma 4.2, we calculate the probability p_c of a closed square and show that it can be made arbitrarily small as $\epsilon \to 0$ and as $a \to \infty$. This will show that the open squares percolate for appropriate a and ϵ.

Lemma 4.2. *The probability p_c of a square being closed can be made arbitrarily small for large enough contraction ratios and large enough a.*

Proof. A square is closed if it satisfies any of the following conditions:

1. It is empty of points, which happens with probability $e^{-\lambda 4a^2} = e^{-4a^2}$;

2. It contains a point whose nearest neighbor is at a distance greater than a. This happens with a probability less than

$$\mathbb{E}[\# \text{ of points in the square}]\mathbb{P}^0[h(0,\Phi) > a] =$$
$$4a^2 \int_a^\infty 2\pi r e^{-r^2\pi} dr = 4a^2 e^{-4a^2\pi}.$$

3. Finally it contains a point which has its nearest neighbor at distance less than a but has a contraction ratio greater than ρ. This probability can

be upper bounded by

$\mathbb{E}$[#of points with nearest neighbor distance less than a and
$\rho > 1-\epsilon$ in a square of side 2a] =
$4a^2 \cdot \mathbb{P}^0[d(0, h(0,\Phi)) < a$ and that the origin has $\rho > 1-\epsilon] =$
$4a^2 \cdot p_0(a,\epsilon)$,

where $p_0(a,\epsilon)$ was calculated in the Lemma 4.1.

Combining them together, we get that the probability that a square is closed, p_c, is less than

$$p_c \leq e^{-4a^2} + 4a^2 e^{-a^2\pi} + a^2 \cdot p_0(a)$$
$$p_c \leq 4e^{-a^2} + 4a^2 e^{-4a^2\pi} + a^2 \cdot \left(\epsilon \cdot \frac{a^2\pi}{\frac{4\pi}{3}+\frac{\sqrt{3}}{2}}\left(e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})} + \frac{4\pi}{3} + \frac{17}{8}\right) + o(\epsilon)\right).$$

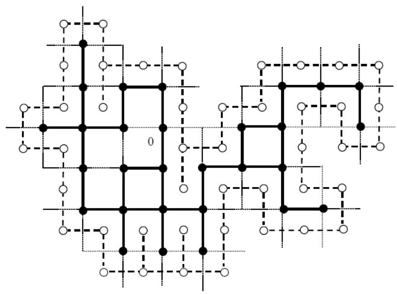
If we take $\epsilon(a) = o\Big(a^4(e^{-a^2(\frac{4\pi}{3}+\frac{\sqrt{3}}{2})})\Big)$, then as $a \to \infty$, p_c goes to 0.

□

Lemma 4.3. *[27, 34] There is a $p_{open} < 1$ such that in any k-dependent site percolation measure on $\mathbb{Z}^2$ satisfying the additional condition that each edge is open with probability at least p_{open}, the probability that $|C_0| = \infty$ is positive.*

We know that $p_{open} = 1 - p_c$, and since p_c can be as small as we need, it follows that there exists p_{open} such that the open squares percolate.

Component 2: Since closed squares do not percolate, there exists a closed circuit of open squares surrounding it [44]. Thus, there exists a closed circuit of open squares around the origin. An infinite forward path from 0 has to cross



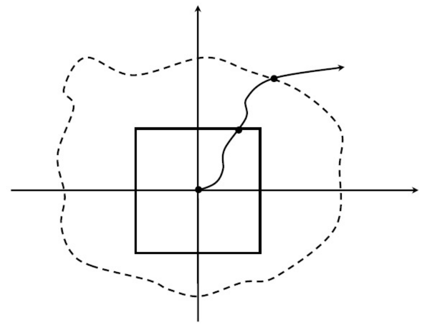


Figure 4.4: Figures from [44]. **Left:** A finite open cluster at the origin, surrounded by a closed circuit in the dual lattice. **Right:** An infinite path from the origin crosses an open square a.s.

this circuit of open squares somewhere. This follows directly from the Konig's lemma [37], which says that an infinite path has to to go infinity. Once it crosses an open square, by construction, all points in the forward set are at a distance less than a from their nearest neighbor. An infinite path without cycles must hence cross infinitely many open squares.

Component 3: In the next part of the proof, we need to study the relations between Palm probabilities corresponding to different random measures or point processes living on the same probability space and being jointly stationarity. We use mainly the Mass Transport Formula (Theorem 6.1.34. in [9]).

We split Φ into two disjoint sub-point processes, $\tilde{\Phi}$ and $\hat{\Phi}$, such that $\Phi = \tilde{\Phi} + \hat{\Phi}$. We call the points of $\tilde{\Phi}$ the "good" points and the points in $\hat{\Phi}$ as the "bad" points.

Let $\tilde{\Phi}$ be the sub-point process of Φ with points with

1. a nearest neighbor at distance at most a,
2. contraction ratio at most ρ.

By construction, open squares contain only "good" points. Thanks to the percolation of open squares, all infinite paths starting from 0 contain infinitely many "good" points.

Points that do not satisfy the criteria for $\tilde{\Phi}$ belong to $\hat{\Phi}$.

For the "good" sub-point process, $\tilde{\Phi}$ we have the following lemma:

Lemma 4.4. *Under the Palm probability of $\tilde{\Phi}$, mean size of the Backward set of 0 under $\tilde{\Phi}$ is integrable* a.s.

Proof. Let us denote the Palm expectation of $\tilde{\Phi}$ by $\tilde{\mathbb{E}}^0$. For each point x of Φ, we send mass $d(h^2(y,\Phi), h(y,\Phi))$ to point y if y is in the forward set of x and $y \in \tilde{\Phi}$. We will apply the mass transport formula with

$$m(x,y,\omega) = d(h^2(y,\Phi), h(y,\Phi))\mathbf{1}\{y \in \tilde{\Phi}, \text{and } y \in For(x,\Phi).\}.$$

Then

$$\lambda\mathbb{E}^0\Big[\int_{\mathbb{R}^2} m(0,y,\omega)\tilde{\Phi}(dy)\Big] = \tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} d(h^2(y,\Phi), h(y,\Phi))\mathbf{1}\{y \in For(0,\Phi)\}\Phi(dy)\Big].$$

Since

$$\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} d(h^2(y,\Phi), h(y,\Phi))\mathbf{1}\{y \in For(0,\Phi)\}\Phi(dy)\Big] < a\sum_{i=0}^{\infty}(1-\epsilon)^i < \infty,$$

it implies that

$$\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(y,0,\omega)\Phi(dy)\Big] < \infty,$$

which implies that the size of the backward set of the origin under $\tilde{\Phi}$ is integrable a.s. Furthermore,

$$\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(y,0,\omega)\Phi(dy)\Big] \geq |Back(0,\tilde{\Phi})| \cdot a,$$

which implies that the size of the backward set of the origin under $\tilde{\Phi}$ is finite a.s.

□

Now we're ready for the final part of the proof.

We claim that the forward path has to be finite a.s. We proved that the number of open squares crossed by the forward path is bounded. Of course, a forward path crosses closed squares as well. So once the infinite path has crossed all its open squares, it can only cross closed squares. In addition, a path can then only go from one closed square to the neighboring closed square. However, the connected component of any closed square is a.s. finite. Therefore there does not exist a closed path that goes to infinity avoiding open squares. It follows that the number of closed squares in the forward path is also finite a.s.. We immediately get the desired result, namely the forward path is finite a.s.

Lemma 4.5. *The Backward set of* 0 *is* a.s. *finite under the Palm probability of* Φ*.*

Proof. Let us denote the Palm expectation of $\hat{\Phi}$ as $\hat{\mathbb{E}}^0$. This proof again uses the mass transport principle in the following way:

$$m(x,y,\omega) = \mathbf{1}\{x \in \hat{\Phi}, \text{and } |Back(x,\hat{\Phi})| = \infty\}\mathbf{1}\{y \in \tilde{\Phi}, y \in For(x,\Phi)\}.$$

$$\tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(0,y,\omega)\hat{\Phi}\}(dx)\Big] =$$
$$\tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} \mathbf{1}\{|Back(x,\hat{\Phi})| = \infty\}\mathbf{1}\{y \in \tilde{\Phi}, y \in For(x,\Phi)\}\hat{\Phi}(dx)\Big]$$
$$\hat{\lambda}\hat{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} \mathbf{1}\{x \in \hat{\Phi}, \text{and } |Back(0,\hat{\Phi})| = \infty\}\mathbf{1}\{y \in For(0,\Phi)\tilde{\Phi}(dy)\Big].$$

Which is the same as

$$\tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(0,y,\omega)\hat{\Phi}(dy)\Big] = \hat{\lambda}\hat{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(x,0,\omega)\tilde{\Phi}(dx)\Big].$$

Hence, the expected mass in, is the expectation under the Palm of $\tilde{\Phi}$ of the number of "bad" points of the predecessors of 0 with an infinite backward set. As a consequence of Lemma 4.4, this expected mass in has to be 0. Hence, the expected mass out is also zero. The Palm probability of $\hat{\Phi}$ that the origin has a finite backward set a.s.

□

Here is an alternative proof of Lemma 4.5 based on the Point-Map cardinality classification [29]. Since the forward path is finite a.s., Follower party has to be $\mathcal{F}/\mathcal{F}$. Hence, the backward path has to be finite a.s.

Therefore we obtained the desired result and completed the alternative proof that the Poisson descending chain is finite a.s.

□

4.2.2 Follower Parties at Step 1

We now extend the proof to the first step of the dynamics.

The Poisson Follower Model at step 1 does not percolate.

The idea of the proof is the same as in the proof of lack of percolation at step 0. At first glance, the proof for step 1 might seem repetitive to the reader. We want to stress that a first essential difference comes in the percolation argument. Since at step 1, the point process is not Poisson, we need to adjust the percolation argument. In addition, it is quite cumbersome to write out all the movements of points and get the probability for some contraction ratio of points of Φ_1 as we did for step 0. Instead, and this is the second important difference, we use a combinatorial argument to show that the probability of having a large enough contraction ratio goes to 0 as the ratio goes to 1. This argument is based on the fact that the factorial moment measure of order 3 of Φ_1 has a bounded density. We use this to show that our new square tessellation percolates. Then, we use again Peierl's argument to show that under the Palm probability of Φ_1, the forward path from the origin has to hit an open square. In this case we have to add extra conditions for a square to be open. We show that the 4-dependent open boxes percolate. Therefore, if the forward set is infinite, it has to cross infinitely many open squares. Finally, we use the mass transport formula to show that the forward and the backward sets of the typical point are a.s.finite.

The contraction ratio is naturally extended to Φ_1,

$$\rho(x) = \frac{d(h(x,\Phi_1), h^2(x,\Phi_1))}{d(x, h(x,\Phi_1))} 1_{h^2(x,\Phi_1)\neq x}. \tag{4.6}$$

For mutually closest neighbors in Φ_1, we again define the contraction ratio to be 0.

Theorem 4.3. *The forward and the backward sets of* 0 *in* Φ_1 *under* $\mathbb{P}^0_{\Phi_1}$ *are* a.s. *finite.*

Proof. The proof is based on a series of lemmas.

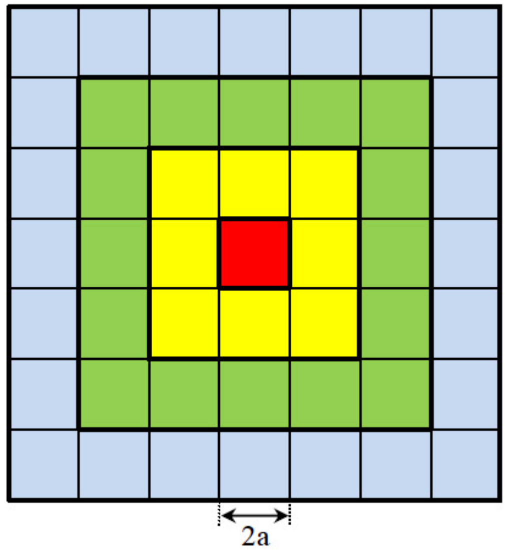


Figure 4.5: Square and its layers. Layer 0 is shown in red, layer 1 is in yellow, layer 2 is in green and layer 3 is in gray

For the square lattice tessellation, we again use a square of side length $2a$. Let $\epsilon > 0$ be the parameter in the contraction ratio.

In this proof we need to look at more than just the square and its neighbors. In order to make the discussion easier, we will call the square in focus, layer 0, the first 8 nearest squares, layer 1, the next 16 nearest squares, layer 2, etc. The different layers are illustrated in Figure 4.5 in different colors. For example, layer 1, which has 8 squares is in yellow.

The definition of open square differs from that at step 0:

We want the open squares under Φ_1 to have the same properties as the open squares at step 0 under Φ_0. That is, the conditions on Φ_1 are:

(i) The square of layer 0 contains at least one point of Φ_1, and each square in layer 1 contains at least one point of Φ_1.

(ii) Each point of Φ_1 in the square of layer 0 has its nearest neighbor at distance at most a, where $2a$ is the side length of the squares. We have the same requirement for points in layer 1.

(iii) Each point in the squares in layers 0 and 1 has a contraction ratio of at most ρ with respect to Φ_1.

These conditions are meant to provide the desired properties, as above, namely any follower path of Φ_1 has to cross open squares and when it does, the path has steps of at most a.

In order to have k-dependency, we add the following conditions on points of Φ_0:

(a) For each square in layers 0 and 1, add an inscribed square in the center with side length a. Each of the inscribed squares should have at least one point of Φ_0. In addition, each point of Φ_0 in the squares of layers 0 and 1 has a nearest neighbor at a distance at most a. See Figure 4.6 for an illustration.

(b) Each square in layers 2 and 3 has at least one point of Φ_0 and all points of Φ_0 in these squares have a nearest neighbor at a distance at most a.

Condition (a) on Φ_0 ensures (i) on Φ_1. Since the distances to the nearest points under Φ_0 in layers $0-4$ are less than a, these distances will remain less than a under Φ_1 in layers 0 and 1 as well. So conditions (a) and (b) on Φ_0 ensure condition (ii) on Φ_1.

A square is *closed* if it not open. The proof of 4-dependence is established in Lemma 4.6.

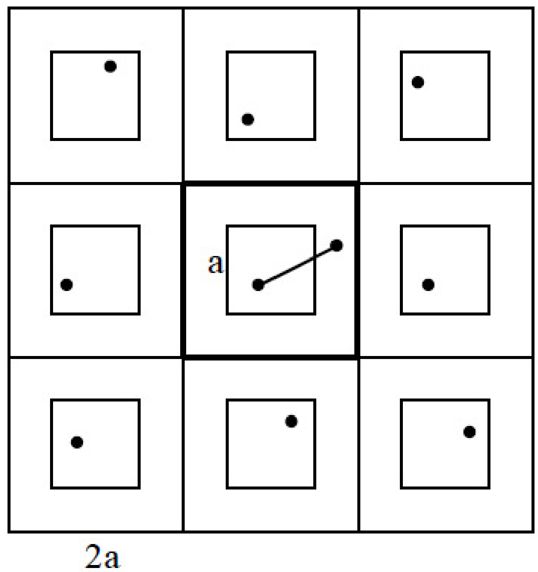


Figure 4.6: Additional conditions on Φ_0 for layer 0 and layer 1. Each inner square in the layer 0 and 1 has at least one point.

Then in Lemma 4.8, we show that the probability that a square is open can be made arbitrarily close to 1 by choosing a and ϵ properly.

We can again break the proof of Theorem 4.3 into three separate components:

1. For a proper choice of lattice edge $2a$, there is a percolation of the open squares;
2. Each infinite path from 0 crosses infinitely many open squares;
3. Use Mass Transport Formula to show that the forward path from 0 has to be finite and then show that the backward set from 0 also has to be finite a.s.

Component 1: Percolation of open squares This is the main difference with the proof of step 0. For step 0, we relied on the independence of the Poisson point process and could argue that there is 3-dependence. Now we show that the states of the squares as defined above are 9-dependent.

Let us call *outer square* the square that contains all the layers of elementary squares up to and including layer 4 and *inner square* the square which contains all elementary squares of layers 0 and 1.

Lemma 4.6. *Under the definitions for the open squares of this section, the site percolation model is 9-dependent.*

Proof. We want to show two things. Namely

(j) The locations of points of Φ_1 in layers 0 and 1 are a function of the points of Φ_0 in layers $0-4$.

(j) The contraction ratios of points of Φ_1 in layers 0 and 1 are a function of the points of Φ_0 in layers $0-4$.

To prove (j) it is enough to prove that no point of Φ_0 outside the outer square penetrates the inner square of step 1.

Lets pick an arbitrary point of Φ_0 which is outside the outer square and at distance l from the outer square. Since every small square in layer 4 has a point of Φ_0, the distance to the nearest point from layer 4, d, is less than $l+2a\sqrt{2}$. This follows directly from the triangle inequality. On the other hand, in order for the same point to enter the inner square, the point has to move at least $l+4a$. This means, it needs to follow a point which is at least $2l+8a$ away. Therefore we get the condition that

$$l+2a\sqrt{2} > 2l+8a,$$

which is impossible.

To prove (jj) is it enough to prove that for all points x of Φ_1 of layers 0 and 1, $h(x, \Phi_1)$ and $h^2(x, \Phi_1)$ are determined by the locations of the points of Φ_0 inside the outer square.

The idea is that the points of Φ_0 in layers 2 and 3 act as a "shield" that prevent points from outside the outer square to enter the inner square at step 1.

Take a point x from layers 0 or 1 in Φ_1. From the condition (ii) on Φ_1 on open squares, we know that its nearest neighbor is at a distance of at most a. Therefore the distance between x and $h^2(x, \Phi_1)$ is less than $2a$. Since the sides of squares have length $2a$, it follows immediately that both $h(x, \Phi_1)$ and $h^2(x, \Phi_1)$ have to be within the layers 0-2. So it is enough to prove that the position of points in Φ_1 in layers $0 - 2$ is determined by the points of Φ_0 in layers $0 - 3$. From the argument above, no point of Φ_0 from outside the outer square, could have entered the 2^{nd} layer in Φ_1. Therefore all the points of Φ_0 that play a role in the contraction ratio of x with respect to Φ_1 are within the outer square.

Therefore, because we want two squares together with their "shield" layers to be disjoint, we need the squares to be at least distance 9 apart. Thus, we have a 9-dependent model.

$\square$

The "shield" idea is inspired in part by the Loop and Shield Conditions of [48]. We calculate the probability that the distance to the nearest point is less than a and that the contraction ratio of the origin is greater than $1 - \epsilon$. Then, we calculate the probability p_{open} of an open square and show that it can be made arbitrarily close to 1 as $\epsilon \to 0$ and as $a \to \infty$. This will show that the open squares percolate.

Lemma 4.7. *The Palm probability for Φ_1 that the distance to the nearest point is less than a and that the contraction ratio of the origin is greater than $1-\epsilon$ tends to 0 as $\epsilon \to 0$.*

Proof. We need to show that as $\epsilon \to 0$ the probability that a contraction ratio of 0 is greater than $(1-\epsilon)$ goes to 0, i.e.,

$$\mathbb{P}^0_{\Phi_1}\Big[(1-\epsilon)d(0,h(0,\Phi_1)) < d(h(0,\Phi_1),h^2(0,\Phi_1))|\, d(0,h(0,\Phi_1)) < a\Big] \to 0, \epsilon \to 0, \tag{4.7}$$

where $\mathbb{P}^0_{\Phi_1}$ denotes the Palm probability under Φ_1. This is enough to prove percolation of good boxes under Φ_1. Let A denote the event $(1-\epsilon)d(0,h(0,\Phi_1)) < d(h(0,\Phi_1),h^2(0,\Phi_1)) < d(0,h(0,\Phi_1)) < a$.

In order to prove Equation (4.7), we use the fact that the factorial moment measure of order 3 of Φ_1 has a bounded density. The argument is combinatorial in nature and we leave proof in the Appendix section A. Therefore,

$$\begin{aligned}
&\mathbb{P}^0(A) = \mathbb{E}[\sum_{X\in B_1} \mathbf{1}_{\theta_x A}] \\
&= \mathbb{E}[\sum_{X\in B_1}\sum_{Y\in\Phi_1} \mathbf{1}_{B(X\to Y)=1} \sum_{Z\in\Phi_1} \mathbf{1}_{B(Y\to Z)=1}\mathbf{1}_{d(X,Y)\le a}\mathbf{1}_{d(Y,Z)\le a}] \\
&\le \mathbb{E}\Big[\sum_{X\in B_1}\sum_{Y\in\Phi_1}\sum_{Z\in\Phi_1} \mathbf{1}_{d(X,Y)\le a}\mathbf{1}_{d(Y,Z)\le a}\mathbf{1}_{(1-\epsilon)d(X,Y)\le d(Y,Z)\le d(X,Z)}\Big] \\
&= |B_1|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \gamma^3(0,y,z)\mathbf{1}_{|y|<a}\mathbf{1}_{|z-y|\le a}\mathbf{1}_{(1-\epsilon)|y|<|z-y|<|y|}dydz \underset{\epsilon\to 0}{\to} 0,
\end{aligned}$$

where γ^3 denotes the 3^{rd} moment density of Φ_1 and B_1 is a ball of volume 1. The relation holds since the volume of the region $\Big((1-\epsilon)|y| < |z-y| < |y|\Big)$ goes to 0 as $\epsilon \to 0$.

Let us denote by $p_{ct}(a)$, the probability that the origin has a contraction ratio less than $1-\epsilon$ under Φ_1^0.

$\square$

Now that we have a relation between ϵ and the probability for a point to have a contraction ratio greater than $1-\epsilon$, we can find the probability p_{open} that the square is open.

Lemma 4.8. *The probability p_{open} of a square being open can be made arbitrarily large for large enough contraction ratios and large enough a.*

Proof. Again, a square is open if it satisfies all of the following conditions:

i For each square in layer 0 and 1, add an inscribed square in the center with an side length a. Each of the inscribed squares should have at least one point and a nearest neighbor at a distance at most a. The probability that the 9 center squares of side length a have at least one point is

$$(1-\mathbb{P}(\text{void probability of a square side length a}))^9 = (1-e^{-\lambda a^2})^9.$$

ii Each point in the square has its nearest neighbor at distance at most a, where $2a$ is the side length of the squares. We have the same requirement for the layers 0, 1, 2 and 3. The probability that one point has its nearest neighbor at a distance at most a is

$$\int_0^a 2\pi r e^{-r^2\pi} dr = 1-e^{-a^2\pi}.$$

In total we have 40 squares, which are all independent of each other in Φ_0.

iii Each point in the square and the neighboring 8 squares has a contraction ratio of at most ρ in Φ_1. This happens with probability greater than

$$\mathbb{E}[\#\text{of points with } \rho < 1-\epsilon \text{ in a square of side 6a}] =$$
$$36a^2 \cdot \mathbb{P}^0[d(0, h(0, \Phi_1)) < 6a, \text{ and that the origin has } \rho < 1-\epsilon] =$$
$$36a^2 \cdot p_{ct}(a, \epsilon),$$

where $p_{ct}(a, \epsilon)$ is the probability that the origin has a contraction ratio of less than $1-\epsilon$ under Φ_1^0.

Combining them together, we get that the probability that a square is open p_{open}, is greater than the product of the probabilities written above. As $\epsilon \to 0$ and $a \to \infty$, $p_{open} \to 1$.

$\square$

Lemma 4.9. *[34, 27] There is a $p_{open} < 1$ such that in any k-dependent site percolation measure on $\mathbb{Z}^2$ satisfying the additional condition that each edge is open with probability at least p_{open}, the probability that $|C_0| = \infty$ is positive.*

We know from Lemma 4.8 that p_{open}, can be as close to 1 as we need, therefore there exists p_{open} such that it percolates. The proof of Lemma 4.9 can be found in [34].

Therefore, if we take p_{open} large enough, it follows from k-dependent percolation that the set of open squares percolates.

$\square$

Component 2: Again, since closed squares do not percolate, there exists a closed circuit of open squares surrounding it [44]. Thus, there exists a closed circuit of open squares around the origin. An infinite forward path from 0 has to cross this circuit of open squares somewhere. Once it crosses an open square, by construction, all points in the forward set are at a distance less than a from their nearest neighbor. An infinite path without cycles must hence cross infinitely many open squares.

Component 3: This part of the proof is also the same as for step 0, just applied to Φ_1. We again study the relations between Palm probabilities corresponding to different random measures or point processes living on the same probability space and being jointly stationarity. We use mainly the Mass Transport formula (Theorem 6.1.34. in [9]).

We split Φ_1 into two disjoint sub-point processes, $\tilde{\Phi}_1$ and $\hat{\Phi}_1$, such that $\Phi_1 = \tilde{\Phi}_1 + \hat{\Phi}_1$. We call the points of $\tilde{\Phi}_1$ the "good" points and the points in $\hat{\Phi}_1$ as the "bad" points.

Let $\tilde{\Phi}_1$ be the sub-point process of Φ_1 with points with

1. a nearest neighbor at distance at most a,

2. contraction ratio in Φ_1 is at most ρ.

By construction, open squares contain only "good" points. Thanks to the percolation of open squares, all infinite paths starting from 0 contain infinitely many "good" points.

Points that do not satisfy the criteria for $\tilde{\Phi}_1$, belong to $\hat{\Phi}_1$.

For the "good" sub-point process, $\tilde{\Phi}_1$ we have the following lemma:

Lemma 4.10. *Under the Palm probability of $\tilde{\Phi}_1$, the mean size of the Backward set of 0 under $\tilde{\Phi}_1$ is finite* a.s.

Proof. Let us denote the Palm expectation of $\tilde{\Phi}_1$ by $\tilde{\mathbb{E}}^0$. For each point x of Φ_1, we send mass $d(h^2(y,\Phi), h(y,\Phi))$ to point y if y is in the forward set of x and $y \in \tilde{\Phi}_1$. We will apply the mass transport formula with

$$m(x,y,\omega) = d(h^2(y,\Phi_1), h(y,\Phi_1))\mathbf{1}\{y \in \tilde{\Phi}_1, \text{and } y \in For(x,\Phi_1).\}.$$

Then

$$\lambda\mathbb{E}^0\Big[\int_{\mathbb{R}^2} m(0,y,\omega)\tilde{\Phi_1}(dy)\Big] = \tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} d(h^2(y,\Phi_1), h(y,\Phi_1))\mathbf{1}\{y \in For(0,\Phi_1)\}\Phi_1(dy)\Big].$$

Since

$$\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} d(h^2(y,\Phi_1), h(y,\Phi_1))\mathbf{1}\{y \in For(0,\Phi_1)\}\Phi_1(dy)\Big] < a\sum_{i=0}^{\infty}(1-\epsilon)^i < \infty,$$

it implies that

$$\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(y,0,\omega)\Phi_1(dy)\Big] < \infty,$$

which means that the size of the backward set of the origin under $\tilde{\Phi}_1$ is integrable a.s.

Additionally, we again have that

$$\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(y,0,\omega)\Phi_1(dy)\Big] \geq |Back(0,\tilde{\Phi}_1)| \cdot a,$$

which implies that the size of the backward set of the origin under $\tilde{\Phi}_1$ is finite a.s.

□

Again, we're ready for the final part of the proof. This part of the proof is the same as for Follower Model at step 0.

We again claim that the forward path has to be finite a.s. We proved that the number of open squares crossed by the forward path is bounded. Of course, a forward path crosses closed squares as well. So once the infinite path has crossed all its open squares, it can only cross closed squares. Additionally, a path can only cross from one closed square to the adjacent closed squares. However, the connected component of closed squares is finite. Therefore there does not exist a closed path that goes to infinity avoiding open squares. It follows that forward path through closed squares is also finite a.s. We immediately get the desired result, namely the forward path is finite a.s.

Lemma 4.11. *The Backward set of* 0 *is* a.s. *finite under the Palm of* Φ_1*.*

Proof. Let us denote the Palm expectation of $\hat{\Phi}_1$ as $\hat{\mathbb{E}}^0$. This proof again uses the mass transport principle in the following way:

$$m(x, y, \omega) = \mathbf{1}\{x \in \hat{\Phi}_1, \text{and } |Back(x, \hat{\Phi_1})| = \infty\}\mathbf{1}\{y \in \tilde{\Phi_1}, y \in For(x, \Phi_1)\}.$$

$$\begin{aligned}
&\tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(0, y, \omega)\hat{\Phi_1}\}(dx)\Big] = \\
&\tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} \mathbf{1}\{|Back(x, \hat{\Phi}_1)| = \infty\}\mathbf{1}\{y \in \tilde{\Phi}_1, y \in For(x, \Phi_1)\}\hat{\Phi}_1(dx)\Big] \\
&\hat{\lambda}\hat{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} \mathbf{1}\{x \in \hat{\Phi}_1, \text{and } |Back(0, \hat{\Phi}_1)| = \infty\}\mathbf{1}\{y \in For(0, \Phi_1)\tilde{\Phi}_1(dy)\Big].
\end{aligned}$$

Which is the same as

$$\tilde{\lambda}\tilde{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(0, y, \omega)\hat{\Phi}_1(dy)\Big] = \hat{\lambda}\hat{\mathbb{E}}^0\Big[\int_{\mathbb{R}^2} m(x, 0, \omega)\tilde{\Phi}_1(dx)\Big].$$

Hence, the expected mass in, is the expectation under the Palm of $\tilde{\Phi}_1$ of the number of "bad" points of the predecessors of 0 with an infinite backward set. As a consequence of the Lemma 4.10, this expected mass in has to be 0. Hence, the expected mass out is also zero. The Palm probability of $\hat{\Phi}_1$ that the origin has a finite backward set a.s.

$\square$

Again, we have an alternative proof of Lemma 4.11 based on the Point-Map cardinality classification [29]. Since the forward path is finite a.s., Follower party has to be $\mathcal{F}/\mathcal{F}$. Hence, the backward path has to be finite a.s. Which is what we wanted to show. $\square$

Therefore, by extending the proof of step 0, we have shown that the Follower dynamics at step 1 does not percolate.

4.2.3 Follower Parties at later steps

We can in principle extend the proof of lack of percolation of follower parties at step 1 to later steps. Comparing the proofs for step 0 and step 1, we see that an essentially different part is the k-dependent percolation argument. We again tessellate the plane with the square grid of side length a and give the same set of conditions for a square to be open. Then, for each step, we need to know from how far away would a point come to an open square. Depending on the step of the dynamics, we can potentially find a k and then apply the k-dependence argument.

In addition, for each step we need to show that the 3^{rd} factorial moment has a bounded density. This can be achieved by a combinatorial argument. The rest of the proof would be the same as for step 0 as it doesn't use Poisson point process and is a general argument.

Chapter 5

Some Geometric Properties of the Follower Graph

"at any rate, there's no harm in trying."

Lewis Carroll, Alice's Adventures in Wonderland

The main aim of this chapter is to discuss some geometric properties of the Follower Graph and certain tessellations associated with it. These properties are of independent interest. In addition, they can potentially be used for an alternative proof of lack of percolation. In Section 5.1, we introduce the Poisson Voronoi tessellation and the Delaunay triangulation. In Section 5.2, we extend the tessellation to the Follower Model, and give bounds on the mean size of these cells. In Section 5.3, we discuss an alternative way to prove the lack of percolation at step 1.

5.1 Introduction

We first define the classical Delaunay triangulation. The reader can look into [46] for further properties. We begin with a set of vertices in $\mathbb{R}^2$ given by a homogeneous Poisson point process Φ_0 whose intensity is assumed to be equal to 1. The Delaunay triangulation $Del(\Phi_0)$ on Φ_0 is the geometric graph on the vertex set Φ_0, with an edge between two vertices $a, b \in \Phi_0$ if and only if there exists a disk whose intersection with the vertex set Φ_0 consists precisely of the points a and b [47].

Furthermore, the *Voronoi cell* of each $a \in \Phi_0$ is defined as

$$\tilde{V}(a, \Phi_0) = \{b \in \mathbb{R}^2 : |b - a| \leq \inf_{X \in \Phi_0} |b - X|\}.$$

This can be extended to Φ_k for all k, when we remove the multiplicity of points. So we have the Voronoi cells of Φ_k and the Delaunay triangulation of Φ_k . This is illustrated in Figure 5.1 for Φ_1.

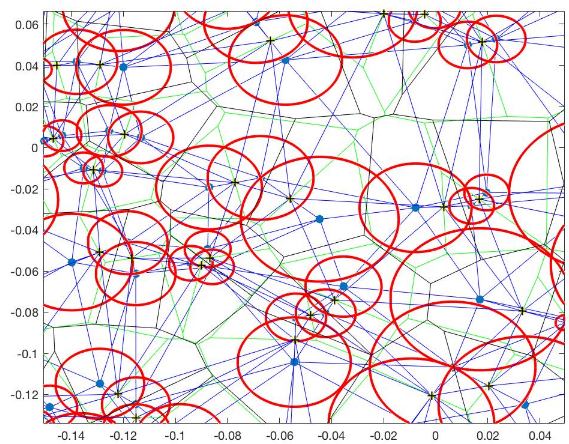


Figure 5.1: Voronoi cells and Delaunay edges after one iteration. In green are the initial Voronoi cells, and in blue after one step. Black points are the positions of points at step 1.

Note that both definitions generalize to higher dimensions as well.

In the following proposition we show that under the follower dynamics at step 1, agents are located at the midway points of certain Delaunay edges of Φ_0.

Proposition 5.1. *Let Φ_0 be the initial stationary simple Poisson point process on $\mathbb{R}^d$. Under the follower dynamics, after one step, Φ_0 becomes Φ_1. Now, denote by $M(\Phi_1)$ the middle points of the Delaunay edges of Φ_0. Then $\Phi_1 \subset M(\Phi_0)$.*

Proof. It was shown in [52] that for all B in Φ_0, if A in Φ_0 is the unique closest neighbor to B, then the line segment AB is an edge in the Delaunay triangulation of Φ_0.

□

Consequently, the action of the Follower Dynamics is a point shift from Φ_0 to $M(\Phi_0)$.

5.2 Voronoi Party Cells and Delaunay Party Triangulation

First, we use the Poisson Voronoi tesselation model as follows. Consider the union of the Voronoi cells that belong to one party. We call this a *Voronoi party cell*. There is one such cell per party. These cells form a tesselation of the Euclidean plane. In this section, we study such Voronoi party cells and their dual, the Delaunay party triangulation.

Definition 5.1. *A party cell of order 0 the union of the Voronoi cells of Φ_0 whose centroids belong to one party at the initial step. Denote a party cell of order zero by $C(x, \Phi_0)$, where x is its centroid and Φ_0 the point process at step 0. We define the centroid of the party cell of order 0 to be the middle of the ultimate leader pair of order 0. Similarly, a party cell of order n is a cell is the union of the Voronoi cells of a party of Φ_n. Points where three Voronoi party cells meet are called extreme points.*

An example of Voronoi party cell of order 0 is shown in Figure 5.3.

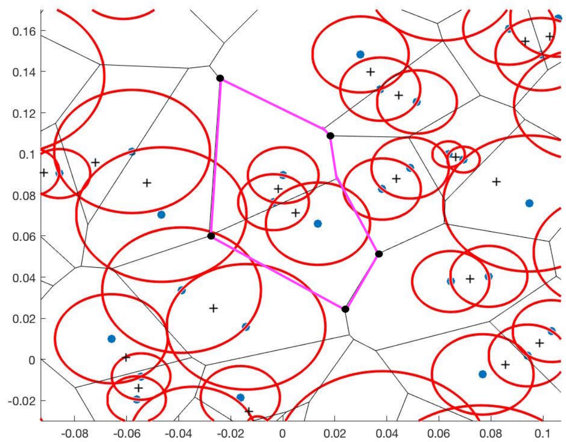


Figure 5.2: Example of a Follower Model at step 0 with its Voronoi cells. The balls feature the empty ball condition defining the leader of each point. One of the Voronoi party cells is labeled in purple and vertices are in black points show where three Voronoi party cells meet.

Definition 5.2. *Let the Delaunay party graph of Φ_0 or party graph for short be defined as follows. The nodes of the party graph are the ultimate leader pairs of rank 0, i.e., the centroids of the party cells of order 0. Call two centroids adjacent if their corresponding party cells share a boundary. Two centroids share an edge only if they are adjacent; the fact that x and y are two adjacent centroids will be represented as $x \sim y$. This defines a random graph on centroids which is the Delaunay party graph. Denote this random graph by $\mathbb{L}$, and write $\mathbb{C}$ for the set of centroids, and $\mathbb{E}$ for the set of edges.*

Below, some geometric properties of the Delaunay party graph are discussed. The main result is that the mean degree of a typical node in the party graph is less than or equal to 6; the proof is given below. We first generalize results of [5] to a specific result on Voronoi party cells.

Proposition 5.2. *Let N_2 be a stationary simple point process on $\mathbb{R}^2$ with intensity $\lambda_2 \in R_+*$. Let N_0 and N_1 be the point processes whose atoms are respectively the vertices and edge centers of the Voronoi Party cells of N_2, and let λ_0 and λ_1 be their respective intensity parameters. Then*

$$2\lambda_1 = \lambda_2 \mathbb{E}_2^0[M],$$

where $\mathbb{E}_2^0$ is the expectation with respect to the Palm probability of the ultimate leader pairs of N_2, and $M := N_1(C(0, N_2))$. If moreover, P-almost surely, there are no 4 points of N_2 lying on a circle of $\mathbb{R}^2$, then

$$3\lambda_0 = \lambda_2 \mathbb{E}_2^0[M].$$

Proof. The proof is using the Mass Transport Principle [6]. Notice that the extreme points of each party cell belong to exactly 3 party cells. Each Delaunay edge of the party cell belongs to the exactly 2 party cells. An example of a party cell and extreme points can be seen in Figure 5.3, where a party cell is colored purple and the extreme points are big black dots.

Define a directed bipartite graph with a directed edge from each $X \in N_2$ to all points of N_1 in the Party Cell $C(x, N_2)$. The constructed graph is translation invariant. So it follows directly from the mass transport formula that $2\lambda_1 \geq \lambda_2 \mathbb{E}_2^0[M]$. Notice that two party cells can share two or more edges. Similarly, one can construct a bipartite graph with a directed edge from each $X \in N_2$ to all points of N_0 in the Party Cell $C(x, N_2)$.

□

Proposition 5.3 (Mean value formulas for planar Voronoi Party Cells.)**.** *Under the conditions of Proposition 5.2, and assuming that almost surely, there* *are no 4 points of N_2 lying on a circle of $\mathbb{R}^2$, then*

$$\mathbb{E}_2^0[M] \leq 6.$$

Proof. We can apply Euler's formula since we have a planar graph in the plane with no edge intersection.

$$v - e + f = 2,$$

where v is the number of vertices, e is the number of edges and f is the number of faces. Then $\lambda_0 - \lambda_1 + \lambda_2 = 0$.

We apply Proposition 5.2 to get

$$\lambda_0 = 1/3\lambda_2\mathbb{E}_2^0[M],$$

$$\lambda_1 = 1/2\lambda_2\mathbb{E}_2^0[M].$$

Since more than one edge can be adjacent to the same Voronoi Party Cells, we get

$$\mathbb{E}_2^0[M] \leq 6.$$

$\square$

As a consequence, the typical Voronoi Party cell has a finite number of neighbors.

5.3 Bond Percolation on the Delaunay Party Graph

We can consider the following (dependent) bond percolation problem on the Delaunay party graph. Say that the edge between two party cells is open if there is a party swap at Φ_1 between them. Each edge will be either open or closed. Note that in order for an infinite follower chain to exist at step 1 we need the Delaunay party graph to percolate (bond percolation) at step 1. Let p denote the probability of having the typical edge be open. We need to first estimate the value of p.

In order to estimate the probability p, we use the calculations obtained for the 4 body swap. Recall that *4 body* swap is a swap that involves 4 different nodes:A, B, A_1, B_1, with, A following B, and A_1 following B_1 at step 0. Then, at step 1, A follows A_1 or A_1 follows A. We estimated this probability in Chapter 3 and obtained a value of $p_{sw} = 0.00012$. The probability p_{sw} is in fact an upper bound for the party swap probability, since every party swap is a part of the 4 body swap.

Let N be the number of points that could be part of a 4 body swap in a given Voronoi party cell. From Neveu's exchange formula [6], the mean of N in a typical Voronoi party cell is

$$\mathbb{E}^o[N] = p_{sw}\mathbb{E}^o[V],$$

where, p_{sw} is the probability of a type 4 swap, V is the volume of a typical Voronoi party cell, and the expectation is under the Palm probability with respect to the ultimate leader pairs of order 0 (i.e., centroids of the party cells). For a process whose ultimate leader pairs or order 0 have intensity λ_1, we have

$$\mathbb{E}^o[V] = \frac{1}{\lambda_1}.$$

Thus we get a bound

$$\mathbb{E}^o[N] < 0.0002.$$

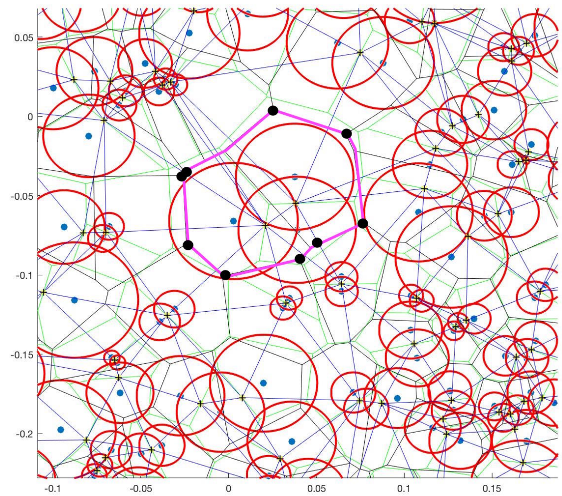


Figure 5.3: Example of a Follower Model at step 1 with its Delaunay triangulation and Voronoi graph. The Delaunay triangulation is shown in blue and Voronoi cells in green. One Voronoi party cell is outlined in purple. Black points show the extreme points.

for $\lambda_1 = 0.62$. Using the fact that the probability for a site to be open is bounded above by the mean of N, we get that $p < 0.0002$.

Conjecture 5.1. *The probability p that a bond is open is below the critical dependent bond percolation threshold on the Delaunay Party graph, and thus the party graph does not percolate at step 1.*

The idea to prove the conjecture is to use arguments similar to the recent paper [51] which establishes a critical treshold for site percolation on the Poisson-Voronoi model. The main difficulty comes from the fact that the graph is random and the bond states are dependent. However, we have good mixing properties and we can expect things to work for the lack of percolation argument. Lack of percolation within this model would provide another proof that the parties are of finite size at step one.

Chapter 6

Limiting Behavior of Party Trees

> ”Go on till you come to the end; then stop.”
>
> Lewis Carroll, Alice’s Adventures in Wonderland and Through the Looking-Glass

This Chapter focuses on the limiting behavior of party trees. Section 6.1 shows some examples of long term dynamics and numerical estimates of the long term distribution. Next, we analyze *stable trees* in Section 6.2. Stable trees are special types of Follower Parties, which are trees for which the graph structure doesn’t change with steps. Such trees have a special structure. As we shall see they have no branching outside the root.

Finally, in Section 6.3 we show some general long term relations. When the number of steps of the Follower Dynamics tends to ∞, there are two subprocesses being formed. One converges in total variation- it is the ultimate leader pair point process, and the other one converges weakly to its limit, the ultimate follower point process. From simulations, in dimension 2, the density of ultimate leader pairs is approximately 0.66. The rest of the points, the ultimate followers, converge weakly to one of the ultimate leader pairs.

6.1 Simulations and Examples of Long-term Behavior

We begin this section by showing some examples observed in the simulation of our dynamics. Each image has its initial chain marked in blue and each point connecting to its closest neighbor also in blue. Red dots signify the positions of the points after 20 time steps. Usually points become very close after that time and in order to observe the pattern, we zoom in the image, and thus the inital blue points are not observed any more. In Figure 6.3 we see an example where there is a follower swap and an agent is following a sibling after one time step. Thus Figure 6.3 is an example of a party restructuring. In Figures 6.5 and 6.6 we see an example of a stable party i.e., the ordering doesn't change. In Figure 6.2 a party has 3 fusion swaps-i.e. we see a party fission.

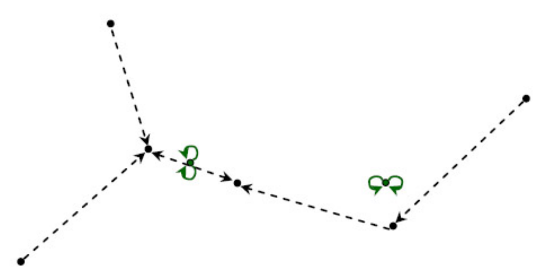


Figure 6.1: Dotted lines show the initial clump and green points position after many steps.

Simulations suggest that in the long term, the typical Follower Party has 3 agents on average.

The 95% confidence interval for the ratio of the number of ultimate leader pairs to total points after 10 steps:$[0.6607, 0.6653]$ for 150 simulations with mean of 1000 points. This ratio is approximately 2/3.

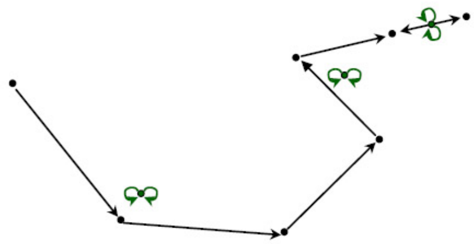


Figure 6.2: Example of a tree that breaks into three ultimate leader pairs-party fission

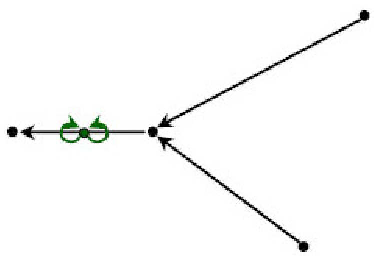


Figure 6.3: Tree at the initial time step.

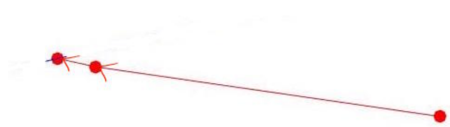


Figure 6.4: Zoomed in situation after 20 time steps

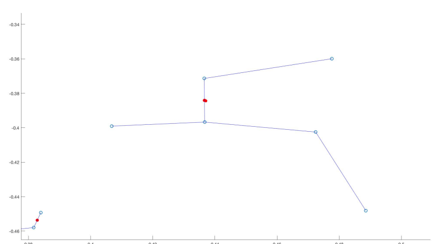


Figure 6.5: Tree at the initial time step.

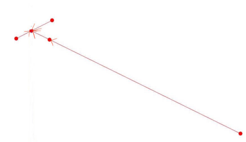


Figure 6.6: Zoomed in situation after 20 time steps

6.2 Stable Tree

We call a party tree *stable* if the leader/followers relationship in this tree does not change over time under the dynamics. An example of a stable tree is shown in Figure 6.5 and Figure 6.6. For such stable parties, we are able to write a closed form trajectory of each agent in the tree based on the location of its ultimate leaders. We will do calculations of positions after the first time step. We will look at one tree of followers/leaders in the following way. Let us denote the position of the ultimate leader pair as $a_1 = a_1^{(0)}$. The opinion of the follower of order 1 will be denoted by $a_2 = a_2^{(0)}$. Now we take a follower of a_2 and denote its opinion as $a_3 = a_3^{(0)}$, etc. The opinion of a follower of order $n-1$ will be denoted as $a_n = a_n^{(0)}$. The position of agent n at time step i will be denoted by $a_n^{(i)}$.

Recall that by our dynamics

$$a_n^{(i)} = \frac{a_n^{(i-1)} + a_{n-1}^{(i-1)}}{2}.$$

Thus we can derive a recursive formula for the position of any agent at any time step that depends only on the positions of the leaders of various order of that agent.

Lemma 6.1. *Under the setting described above, the position of agent a_n at time step i can be written as*

$$a_n^{(i)} = \begin{cases} \frac{1}{2^i} \sum_{k=0}^{i} \binom{i}{k} a_{n-k} & i < n, \\ \frac{1}{2^i} \left(\sum_{k=0}^{n-1} \binom{i}{k} a_{n-k} + \sum_{k=n}^{i} \binom{i}{k} a_1 \right) & i \geq n. \end{cases}$$

Proof. Follows by induction.

□

Note that in the limit, $i \to \infty$, that $\frac{1}{2^i} \sum_{k=0}^{n} \binom{i}{k} a_{n-k} \to 0$ and
$\frac{1}{2^i} \sum_{k=n}^{i} \binom{i}{k} a_1 \to a_1$.

Proposition 6.1. *If the tree is stable, it cannot have branching outside of the root.*

Proof. The proof is by contradiction. Let us take one branch of the tree and look for the first branching off, denoting the point as a_n. Then its two followers are denoted as $a_{n+1,1}$ and $a_{n+1,2}$. Notice that from the Lemma 6.1, the formula for the position of $a^{(i)}_{n+1,1}$, of agent $(n+1,1)$ and of $a^{(i)}_{n+1,2}$, of agent $(n+1,2)$ at time i are

$$a^{(i)}_{n+1,1} = \frac{1}{2^i}\Big(a_{n+1,1} + \sum_{k=1}^{n}\binom{i}{k}a_{n+1-k} + \sum_{k=n+1}^{i}\binom{i}{k}a_1\Big)$$

$$a^{(i)}_{n+1,2} = \frac{1}{2^i}\Big(a_{n+1,2} + \sum_{k=1}^{n}\binom{i}{k}a_{n+1-k} + \sum_{k=n+1}^{i}\binom{i}{k}a_1\Big).$$

So the distance between them $d(a^{(i)}_{n+1,1}, a^{(i)}_{n+1,2})$ is

$$d(a^{(i)}_{n+1,1}, a^{(i)}_{n+1,2}) = \frac{1}{2^i}d(a_{n+1,1}, a_{n+1,2}).$$

Thus the distance between $a_{n+1,1}$ and $a_{n+1,2}$ decreases exponentially with time.

First, we prove the result in the special case when $n = 2$ and then prove in general with the same idea but more calculations.

Position of a_2 changes with time step i in the following way

$$a^{(i)}_2 = \frac{a_2 + (2^i - 1)a_1}{2^i},$$

while $a_{3,1}$ and $a_{3,2}$ move as

$$a^{(i)}_{3,1} = \frac{a_{3,1} + ia_2 + (2^i - i - 1)a_1}{2^i}$$

$$a_{3,2}^{(i)} = \frac{a_{3,2} + ia_2 + (2^i - i - 1)a_1}{2^i}$$

We want to show that there exists i for which $d(a_{3,1}^{(i)}, a_2^{(i)}) > d(a_{3,1}^{(i)}, a_{3,2}^{(i)})$ or $d(a_{3,2}^{(i)}, a_2^{(i)}) > d(a_{3,1}^{(i)}, a_{3,2}^{(i)})$.

Note that

$$d(a_{3,1}^{(i)}, a_2^{(i)}) = \frac{|a_{3,1} + (i-1)a_2 - a_1|}{2^i}.$$

So the question is, whether there is an i such that

$$\frac{|a_{3,1} + (i-1)a_2 - a_1|}{2^i} > \frac{d(a_{3,1}, a_{3,2})}{2^i},$$

or after multiplying by 2^i

$$|a_{3,1} + (i-1)a_2 - a_1| > d(a_{3,1}, a_{3,2}).$$

We will denote the x-coordinates of a_n as $a_{n,1}$ and y-coordinates as $a_{n,2}$ respectively. Thus

$$|a_{3,1} + (i-1)a_2 - a_1| = |(a_{3,1,1} + (i-1)a_{2,1} - a_{1,1}, a_{3,1,2} + (i-1)a_{2,2} - a_{1,2})|.$$

So after squaring the inequality, the question is, whether there is an i such that

$$(a_{3,1,1} - a_{1,1} + (i-1)a_{2,1})^2 + (a_{3,1,2} - a_{1,2}) + (i-1)a_{2,2})^2 > d(a_{3,1}, a_{3,2})^2.$$

After rearranging the terms and grouping them together, we get an inequality of the form

$$(i-1)^2(a_{2,1}^2 + a_{2,2}^2) + (i-1)b + c > 0,$$

where $b = 2((x_{3,1,1} - x_{1,1})x_{2,1} - (x_{3,1,2} - x_{1,2})x_{2,2})$, and $c = (x_{3,1,1} - x_{1,1})^2 + (x_{3,1,2} - x_{1,2})^2 - d(x_{3,1}, x_{3,2})^2$.

Since it is a quadratic polynomial with positive coefficient $(a_{2,1}^2 + a_{2,2}^2 > 0)$, it follows directly that it will be positive for some i, which is what we wanted to show.

Now in a similar way we extend this result to an arbitrary n. We want to show that there exists i such that

$$d(a_{n+1,1}^{(i)}, a_n^{(i)}) > d(a_{n+1,1}^{(i)}, a_{n+1,2}^{(i)}).$$

Recall that for $i > n$

$$a_n^{(i)} = \frac{1}{2^i}\Big(\sum_{k=0}^{n-1}\binom{i}{k}a_{n-k} + \sum_{k=n}^{i}\binom{i}{k}a_1\Big),$$

and

$$a_{n+1,1}^{(i)} = \frac{1}{2^i}\Big(\sum_{k=0}^{n}\binom{i}{k}a_{n+1-k} + \sum_{k=n+1}^{i}\binom{i}{k}a_1\Big).$$

From the initial formulas, we know that the distance changes as

$$d(a_{n+1,1}^{(i)}, a_n^{(i)}) = \Big|\frac{(\sum_{k=0}^{n}\binom{i}{k}a_{n+1-k} + \sum_{k=n+1}^{i}\binom{i}{k}a_1)}{2^i} - \frac{(\sum_{k=0}^{n-1}\binom{i}{k}a_{n-k} + \sum_{k=n}^{i}\binom{i}{k}a_1)}{2^i}\Big|. \tag{6.1}$$

Looking at the left hand side we see the summation

$$\Big|\frac{a_{n+1,1} + \binom{i}{1}a_n + \binom{i}{2}a_{n-1} + ... + \binom{i}{n}a_1 + \binom{i}{n+1}a_1 + ... + \binom{i}{i}a_1}{2^i} - \frac{(\binom{i}{0}a_n + \binom{i}{1}a_{n-1} + ... + \binom{i}{n-1}a_1 + ... + \binom{i}{i}a_1)}{2^i}\Big|. \tag{6.2}$$

Many terms with a_1 cancel and the equation can be written compactly as

$$d(a_{n+1,1}^{(i)}, a_n^{(i)}) = \frac{\Big|\sum_{k=0}^{n-2}\binom{i}{k}(a_{n+1-k} - x_{n-k}) - \binom{i}{n-1}a_1\Big|}{2^i}.$$

The distance between $a^{(i)}_{n+1,1}$ and $a^{(i)}_{n+1,2}$ changes as

$$d(a^{(i)}_{n+1,1}, a^{(i)}_{n+1,2}) = \frac{d(a_{n+1,1}, a_{n+1,2})}{2^i}.$$

Thus we want to show that there exists a time step i such that

$$\Big|\binom{i}{n-1}a_1 - \sum_{k=0}^{n-2}\binom{i}{k}(a_{n+1-k} - a_{n-k})\Big| > d(a_{n+1,1}, a_{n+1,2}).$$

Each of the $\binom{i}{k}$ is a polynomial of i. What we want is to somehow select a highest order polynomial. Take i to be at least twice as n, i.e., $i > 2n$. In that case, the highest degree polynomial will be for $\binom{i}{n-1}$ and is a polynomial of degree $n-1$ with positive coefficient next to i^{n-1} term. We will denote the polynomial $\binom{i}{n-1}a_1$ as $P_{n-1}(i)$, and $-\sum_{k=0}^{n-2}\binom{i}{k}(a_{n+1-k} - a_{n-k})$ as $Q(i)$. Note that $Q(i)$ is a polynomial of order less than $n-1$.

Like in the case $n = 2$ we will denote the x-coordinates as $P_{n-1,1}(i)$ and $Q_1(i)$, and the y-coordinates as $P_{n-1,2}(i)$ and $Q_2(i)$, respectively. So now the inequality we are trying to show can be written more compactly as

$$|(P_{n-1,1}(i) + Q_1(i), P_{n-1,2}(i) + Q_2(i))| > d(a_{n+1,1}, a_{n+1,2}).$$

Again square everything to get

$$(P_{n-1,1}(i) + Q_1(i))^2 + (P_{n-1,2}(i) + Q_2(i))^2 > d(a_{n+1,1}, a_{n+1,2})^2.$$

After multiplying and rearranging we have

$$(P_{n-1,1}(i)^2 + P_{n-1,2}(i)^2) + R > 0,$$

where

$$R = 2(P_{n-1,1}(i)Q_1(i) + P_{n-1,2}(i)Q_2(i)) + (Q_1(i)^2 + Q_2(i)^2 - d(a_{n+1,1}, a_{n+1,2})^2).$$

R is a polynomial of order less than $2(n-1)$ and $P_{n-1,1}(i)^2$ and $P_{n-1,2}(i)^2$ are both polynomials of order $2(n-1)$. Thus we can factor the inequality

$$i^{2(n-1)}\Big(\frac{P_{n-1,1}(i)^2 + P_{n-1,2}(i)^2) + R}{i^{2(n-1)}}\Big) > 0,$$

i.e.

$$i^{2(n-1)}\Big(\frac{P_{n-1,1}(i)^2}{i^{2(n-1)}} + \frac{P_{n-1,2}(i)^2}{i^{2(n-1)}} + \frac{R}{i^{2(n-1)}}\Big) > 0. \tag{6.3}$$

Notice that in Equation (6.3), the first two terms tend to constants as $n \to \infty$, and the last term tends to 0. Therefore the equation on the left hand side is greater than 0, for i large enough. Thus for every n there exists an i such that the distance between $a^{(i)}_{n+1,1}$ and $a^{(i)}_{n+1,2}$ becomes smaller than the distance between $a^{(i)}_{n+1,1}$ and $a^{(i)}_{n}$. Therefore, chain has to break at some point if it has branching outside of the root.

□

Another consequence of this proposition is that, long term, there cannot be branching outside of the root in any tree.

Conjecture 6.1. *For every ultimate leader pair, there exists a step k after which the party tree of the ultimate leader pair is stable.*

6.3 Long Term Relations

In this subsection we look at some mass transport relations that hold in general in the limiting process. Assuming that the limiting objects consist of ultimate leader pairs and ultimate followers, we can derive a relation between them. Let $\lambda = 1$ be the intensity of the initial Poisson point process. Let λ_f be the intensity of the ultimate followers and λ_l be the intensity of the ultimate leaders. Take $\bar{N}$ to be the mean number of ultimate followers of one ultimate leader. Then by the mass transport principle it directly follows that

$$1 \cdot \lambda_f = \bar{N}\lambda_l,$$

using that $\lambda_f + \lambda_l = 1$, implies that

$$\bar{N} = \frac{1}{\lambda_l} - 1.$$

Since there are two ultimate leaders in a pair, the total number of ultimate followers is $N = 2\bar{N}$. This relation should hold regardless of the dimensions.

Now if we plug in $\lambda_l = \frac{2}{3}$, like obtained in the simulations, we get that $N = 1$. Implying that the mean size of a party is 3 in the limit, which is also consistent with the simulations.

Chapter 7

Future steps

> I almost wish I hadn't gone down that rabbit-hole and yet, and yet, it's rather curious, you know, this sort of life!
>
> ---
>
> Lewis Carroll, Alice's Adventures in Wonderland

This chapter is split into two main sections. In Section 7.1 we give a summary and discuss future steps on the Follower Dynamics. In Section 7.2 we discuss the first observations from our explorations of the Hegselmann-Krause Dynamics.

7.1 Summary and Future Steps on the Follower Dynamics

To summarize, we introduced a new model inspired by the problems in opinion dynamics. We described different phenomena that pertain to this dynamics and explored the long term behavior of this system. We gave a new way of calculating frequencies of certain configurations, such as densities of ultimate leader pairs of order zero and one. This technique also gives a way of calculating the probability of an agent leaving its party. Furthermore, we used ideas from lattice percolation to show finiteness of parties at step 0 and 1. We also looked at modified Voronoi tessellations and proved different properties.

In addition, we analyzed the long term behavior of parties and got general results for the limiting follower party shapes.

For future work we hope to prove that the Follower parties are of finite size at all steps. Which is substantiated by simulations. Additionally, we would like to give an explanation to the observed ratio between the densities of the ultimate leader pair point process and the ultimate follower point process. We also hope to extend this result to higher dimensions.

As another branch of future work, we hope to explore the Hegselmann-Krause dynamics of the Poisson point process.

7.2 The Hegselmann-Krause Dynamics - First Observations

This section is divided into two parts. First we introduce the Hegselmann-Krause Dynamics in the particle systems settings. In Section 7.2.2 some instances of the Hegselmann-Krause Dynamics are shown. These cases illustrate some of the difficulties with analyzing this model.

We start with a stationary Poisson Point process Φ in $\mathbb{R}^d$ with intensity λ and hence an infinite number of points. For each compact subset of $\mathbb{R}^2$, the number of points is finite almost surely. The initial point process is simple, since a.s. $\Phi(x) \leq 1$ for each $x \in \Phi$. For each point $x \in \Phi$ let set N_x denote the set of neighbors of x, where a point y is a neighbor of x if $||x - y|| < 1$ (usual Euclidean norm). Note that here we assume that each point is its own neighbor. Let $\tilde{x}$ be the new position of x, which we calculate as $\tilde{x} = \frac{\sum_{y \in N_x} y}{|N_x|}$. Averaging is done simultaneously for all the points. We refer to this averaging as the Hegselmann-Krause model [17]. We repeat the same process many times. Substantiated by simulations, we conjecture that there exists a low enough initial intensity λ_c such that for $\lambda < \lambda_c$ this process converges in the vague topology almost surely. Meaning that restricted to a compact subset $C \subset \mathbb{R}^d$, the process converges and has no accumulations.

Lemma 7.1. *(Percolation threshold and existence of infinite connected component) For each point x of our process Φ, take a ball of radius 1 around it, i.e. $B_1(x)$. Balls that intersect create a connected component of the point process. We state the result proved in [13]: "there exists a nontrivial λ_p such that for $\lambda > \lambda_p$ there exists an infinite connected component". Such regime is called super-critical. If $\lambda < \lambda_p$ we are in the sub-critical regime, and all a connected components are finite.*

This model is referred to as the spherical Boolean model. Note that for now we are only looking at the sub-critical regime.

7.2.1 Collision of Convex Hulls of Poisson Trees

Definition 7.1. *(Convex clumps) We start from a sub-critical Poisson Point process Φ. The convexified clump of a point x of Φ is the convex hull of the clump containing x.*

The reason we are interested in convexified clumps and not just clumps themselves is that there exist situations where disjoint clumps can merge together like in Figure 7.1. Even though at step 0, one ball is disjoint from the other two, after one step of the averaging takes place, the two other balls move to one position where they intersect the previously disjoint ball. So lack of percolation at step 0 does not immediately imply the lack of percolation at step 1.

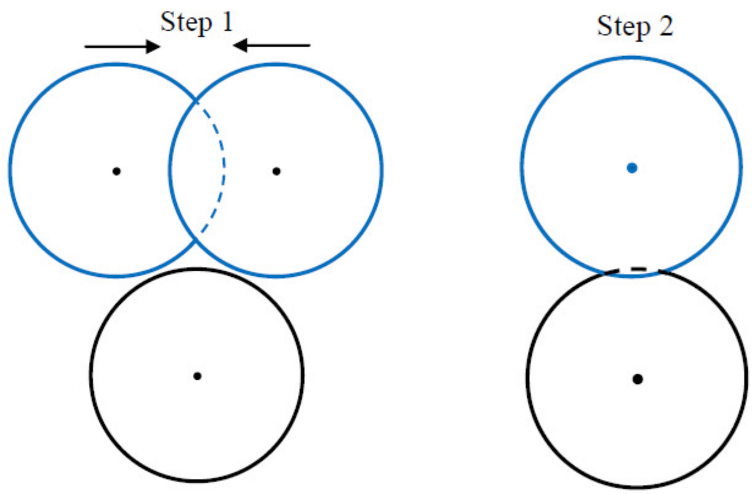


Figure 7.1: Pairwise interaction. Left image is the initial configuration and right image is after one averaging.

7.2.2 Some Instances of Hegselmann-Krause Dynamics

This section is devoted to examples of configurations which illustrate the difficulties that arise when analyzing Hegselmann-Krause Dynamics. The main issue is that agents can travel "large distances" over time steps. By large distance we mean distance of order of the number of time steps. Since there is a chance that an agent travels large distances, we cannot guarantee that the position of an agent A remains fixed after some time, since there can be a time step when a new agent C enters the ball $B(A, 1)$.

One peculiar case we noticed is when we put evenly spaced points on the circle and apply the Hegselmann-Krause Dynamics. Then we get that all points converge to the center of the circle. In Figure 7.2 we have 100 points evenly spaced around the circle of radius 7. It shows the initial positions of the points as open blue circles and after 200 time steps they all end up in the center of the circle, which is shown in red.

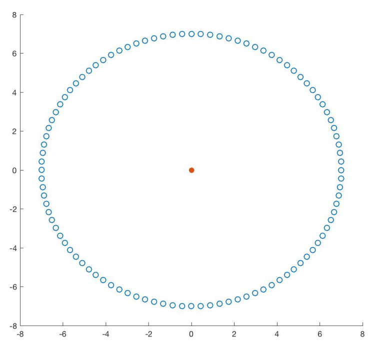


Figure 7.2: Circle with evenly spaced points. Open blue circles are the initial positions. The final position after 200 time steps is in red.

We wonder whether there exist configurations such that we can end up with an accumulation of points. We came up with an idea that if we have concentric circles with points evenly spaced out and inner ones get to the center faster, then the center would be an accumulation point.

If two concentric circles have radius difference for example one, then they start merging before approaching the center. This situation is shown in the left part of Figure 7.3. We have two circles of radius 5 and 6 respectively which are depicted by empty blue circles. After 50 time steps they end up in red circle positions.

However if the distance between two concentric circles is large enough, we claim that they eventually end up in the center of the circles. We show in the right part of Figure 7.3 that indeed the smaller circle converges faster than the larger. However they both end up in the center as shown in left Figure 7.4. We also observed that, with three circles, the situation is the same as in the right part of Figure 7.4. We have three circles with radius 6, 8 and 10 and with the same center. After 400 time steps they all end up in the center of

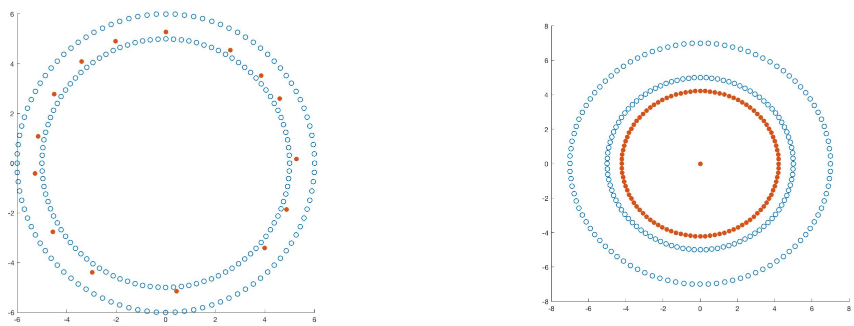


Figure 7.3: **Left:** Initially we have two circles, one of radius 5 other of radius 6 with the same center. They are shown in blue open circles. After 50 time steps they end up in red circles. **Right:** Initially we have two circles, one of radius 5 other of radius 6. They are depicted by blue open circles. After 50 time steps, they end up in red circles.

the circles. Thus we generalize that for any number of circles, as long as the difference between two consecutive radii is sufficient that they will end up in the center.

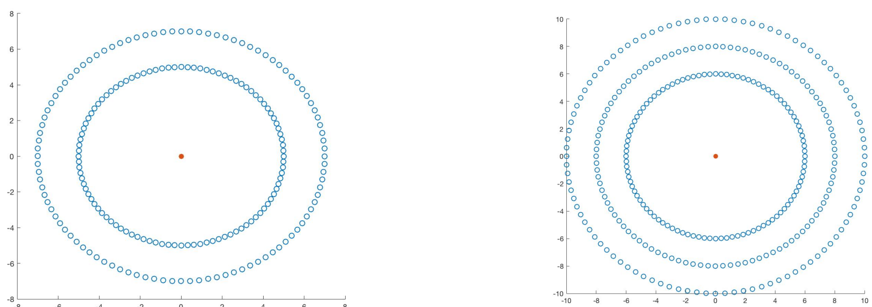


Figure 7.4: **Left:** Two circles of radii 5 and 7. After 200 time steps both circles end up at the center. **Right:** Three circles that have radii 6, 8 and 10. After 400 time steps they all end up in the center.

Appendices

Appendix A

Additional proofs

Here we list some well known proofs and properties.

A.1 Density of Mutually Closest Neighbor Pairs

Lemma A.1. *[14] In a PPP over $\mathbb{R}^2$ with density λ, the density of the mutually closest neighbor pairs is $\theta\lambda$, where $\theta = \frac{3\pi}{8\pi+3\sqrt{3}}$.*

Proof. Suppose that agent A follows agent B. Denote by $P_{fs}^{(1)} = 2\theta$ the probability that B follows A. Density to the nearest neighbor for Poisson point process is $f(r) = 2\lambda\pi r e^{-\lambda\pi r^2}$. If distance between agents A and B is r, it means that the balls centered at A and B with radius r are empty of other points like in Figure A.1. Area of the non-intersecting part of the ball around

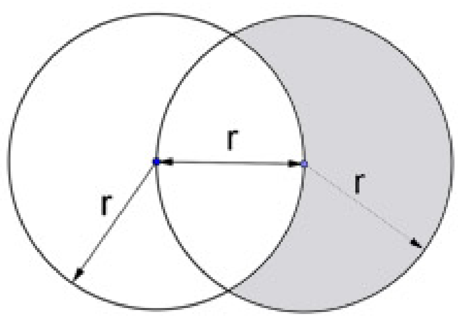


Figure A.1: Mutually closest pair of points

B (shaded area in Figure A.1) is

$$r^2\pi - A_{intersection} = r^2(\pi/3 + \sqrt{3}/2).$$

Where the area of intersection of two circles of same radius is $A_{intersection} = r^2(2\pi/3 - \sqrt{3}/2)$. (Follows from the formula in the chapter below) Thus combining both events we get an integral geometric formula for the probability of having mutually closest neighbor pair

$$P_{fs}^{(1)} = \int_0^\infty e^{-r^2(\pi/3+\sqrt{3}/2)} 2\pi r e^{-r^2\pi} dr = \frac{6\pi}{8\pi + 3\sqrt{3}} \approx 0.62.$$

□

Lemma A.2. *[14] In a PPP over $\mathbb{R}^3$ with density $\lambda = 1$, the density of the mutually closest neighbor pairs is $\theta\lambda$, where $\theta = \frac{8}{27}$.*

Proof. Suppose that agent A follows agent B. Denote by $P_{fs}^{(1)} = 2\theta$ the probability that B follows A. Density to the nearest neighbor for Poisson point process is $f(r) = \frac{4}{3}\pi r^2 e^{-\frac{4}{3}\pi r^3}$. If distance between agents A and B is r, it means that the balls centered at A and B with radius r are empty of other points. Area of the non-intersecting part of the ball around B is

$$\frac{4}{3} r^3 \pi - A_{intersection} = \frac{11}{12} r^3 \pi.$$

Where the area of intersection of two balls of same radius is $A_{intersection} = \pi r(r^2 - r^2/12) = \frac{11 r^3 \pi}{12}$. Thus combining both events we get an integral geometric formula for the probability of having mutually closest neighbor pair

$$P_{fs}^{(1)} = \int_0^\infty e^{-\frac{11}{12} r^3 \pi} 4\pi r^2 e^{-\frac{4}{3} r^3 \pi} dr = \frac{16}{27} \approx 0.59.$$

□

A.2 Higher Order Moment Measures

We first start with the moment measure of order 2 of Φ_1 and then will extend the result to the third factorial moment measure.

Second Order Moment Measure

Lemma A.3. Φ_1 *has a second factorial moment measure with a bounded density.*

Proof. We want to show that

$$\mathbb{E}\Big[\sum_{x_i \neq x_j \in \tilde{\Phi}_1} f(x_i, x_j)\Big] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y)g(x,y)dxdy,$$

with $g(x,u)$ being the density of the 2^{nd} factorial moment measure.

By $\tilde{\Phi}_1$, we denote the point process of Φ at step 1 without the multiplicity. Let $\tilde{\Phi}$ be the Poison point process where one removes the leftmost point of each ultimate leader pair of order 0. There is a bijection between $\tilde{\Phi}$ and $\tilde{\Phi}_1$.

We analyze different possible configurations and show that there is a density of the second moment measure in each disjoint case.

1. Pairs of points of $\tilde{\Phi}$ that follow yet another point, which is the same case as pairs of points of Φ that follow yet another point.

$$\mathbb{E}\Big[\sum_{X_i, X_j \in \tilde{\Phi}, i \neq j}\sum_{X_k \in \Phi} f(\frac{X_i + X_k}{2}, \frac{X_j + X_k}{2})\mathbf{1}_{\Phi(B(X_i \to X_k))=1}\mathbf{1}_{\Phi(B(X_j \to X_k))=1}\Big]$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+z}{2}, \frac{y+z}{2})e^{-\lambda(B(x\to z)\cup B(y \to z))}dxdydz,$$

where we use the that $\tilde{\Phi} = \Phi$ and is a Poisson point process. Applying the change of variable $a = \frac{x+z}{2}$, $b = \frac{y+z}{2}$ and z fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 4f\Big(a,b\Big)\int_{\mathbb{R}^2} e^{-\lambda(B(2a-z\to z)\cup B(2b-z\to z))}dz\,dadb,$$

where $g(a,b) = \int_{\mathbb{R}^2} e^{-\lambda(B(2a-z\to z)\cup B(2b-z\to z))}dz$.

2. Pairs of points of $\tilde{\Phi}$ such that $X_i \to X_j$, $X_j \to X_k \in \Phi$, such that

   (a) $X_k \not\to X_j$. Then we have

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi, i\neq j\neq k} f\Big(\frac{X_i+X_j}{2},\frac{X_j+X_k}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_j))=1}$$
$$\mathbf{1}_{\Phi(B(X_j\to X_k))=1}\mathbf{1}_{\Phi(B(X_k\to X_j)>1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2},\frac{y+z}{2})e^{-\lambda(|B(x\to y)\cup B(y\to z)|)}$$
$$(1-e^{-\lambda(B(z\to y)\setminus B(y\to z))})dxdydz.$$

   Applying the change of variable $a = \frac{x+y}{2}$, $b = \frac{y+z}{2}$ and z fixed, (thus $x = 2a - 2b + z$, $y = 2b - z$) we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 4f\Big(\frac{a}{2},\frac{b}{2}\Big)$$
$$\int_{\mathbb{R}^2} e^{-\lambda(B(2a-2b+z\to 2b-z)\cup B(2b-z\to z))}\Big(1-e^{-\lambda(B(z\to 2b-z)\setminus B(2b-z\to z))}\Big)dz\,dadb,$$

   where

$$g(a,b) = \int_{\mathbb{R}^2} e^{-\lambda(B(2a-2b+z\to 2b-z)\cup B(2b-z\to z))}\Big(1-e^{-\lambda(B(z\to 2b-z)\setminus B(2b-z\to z))}\Big)dz.$$

(b) $X_k \to X_j$, but X_j is left most in (X_j, X_k). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi, i\neq j\neq k} f\Big(\frac{X_i+X_j}{2}, \frac{X_j+X_k}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_j))=1}$$
$$\mathbf{1}_{\Phi(B(X_j\to X_k))=1}\mathbf{1}_{\Phi(B(X_k\to X_j)=1}\mathbf{1}_{X_j^1<X_k^1}\Big]$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2}, \frac{y+z}{2})e^{-\lambda(|B(x\to y)\cup B(y\to z)\cup B(z\to y|)}$$
$$\mathbf{1}_{y_1<z_1}dxdydz.$$

Applying the change of variable $a = \frac{x+y}{2}$, $b = \frac{y+z}{2}$ and z fixed, (thus $x = 2a - 2b + z$, $y = 2b - z$) we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 4f\Big(a, b\Big)$$
$$\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+z\to 2b-z)\cup B(2b-z\to z)\cup B(z\to 2b-z)|)}dz\, dadb,$$

where

$$g(a,b) = \int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+z\to 2b-z)\cup B(2b-z\to z)\cup B(z\to 2b-z)|)}dz.$$

3. In this case we look at the pairs of points such that $X_i, X_j \in \tilde{\Phi}$, $X_i \to X_k$, and $X_j \to X_l$, where $k \neq l$. Then we have the following disjoint cases

(a) $X_k \not\to X_i$ and $X_l \not\to X_j$. Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k,X_l\in\Phi, i\neq j\neq k\neq l} f\Big(\frac{X_i+X_k}{2}, \frac{X_j+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_k))=1}$$
$$\mathbf{1}_{\Phi(B(X_j\to X_l))=1}\mathbf{1}_{\Phi(B(X_k\to X_i)>1}\mathbf{1}_{\Phi(B(X_l\to X_j)>1}\Big]$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\Big(\frac{x+z}{2}, \frac{y+u}{2}\Big)e^{-\lambda(|B(x\to z)\cup B(y\to u)|)}$$
$$\Big(1 - e^{-\lambda(B(z\to x)\setminus B(x\to z))}\Big)\Big(1 - e^{-\lambda(B(u\to y)\setminus B(y\to u))}\Big)dxdydzdu.$$

Applying the change of variable $a = \frac{x+z}{2}$, $b = \frac{y+u}{2}$, with u and z fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f\Big(a,b\Big)$$
$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(a-z\to z)\cup B(b-u\to u)|)}\Big(1-e^{-\lambda(B(z\to a-z)\setminus B(a-z\to z))}\Big)$$
$$\Big(1-e^{-\lambda(B(u\to b-u)\setminus B(b-u\to u))}\Big)dzdu\,dadb,$$

where

$$g(a,b) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(a-z\to z)\cup B(b-u\to u)|)}\Big(1-e^{-\lambda(B(z\to a-z)\setminus B(a-z\to z))}\Big)$$
$$\Big(1-e^{-\lambda(B(u\to b-u)\setminus B(b-u\to u))}\Big)dzdu.$$

(b) $X_k \to X_i$, but X_i is left most in (X_i, X_k), and $X_l \not\to X_j$. Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k,X_l\in\Phi, i\neq j\neq k\neq l} f\Big(\frac{X_i+X_k}{2},\frac{X_j+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_k))=1}$$
$$\mathbf{1}_{\Phi(B(X_k\to X_i))=1}\mathbf{1}_{\Phi(B(X_j\to X_l)=1}\mathbf{1}_{\Phi(B(X_l\to X_j))>1}\mathbf{1}_{X_i^1<X_k^1}\Big]$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+z}{2},\frac{y+u}{2})e^{-\lambda(|B(x\to z)\cup B(z\to x)\cup B(y\to u)|)}$$
$$(1-e^{-\lambda(|B(u\to y)\setminus B(y\to u)|)})\mathbf{1}_{x_1<z_1}dxdydzdu.$$

Applying the change of variable $a = \frac{x+z}{2}$, $b = \frac{y+u}{2}$, with u and z fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 4f\Big(a,b\Big)$$
$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-z\to z)\cup B(z\to 2a-z)\cup B(2b-u\to u)|)}$$
$$(1-e^{-\lambda(|B(u\to 2b-u)\setminus B(2b-u\to u)|)})\mathbf{1}_{2a_1-z_1<z_1}dzdu\,dadb,$$

where

$$g(a,b) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-z\to z)\cup B(z\to 2a-z)\cup B(2b-u\to u)|)}$$
$$(1-e^{-\lambda(|B(u\to 2b-u)\setminus B(2b-u\to u)|)})\mathbf{1}_{2a_1-z_1<z_1}dzdu.$$

(c) $X_k \not\to X_i$ and $X_l \to X_j$, but X_j is left most in (X_j, X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k,X_l\in\Phi, i\neq j\neq k\neq l} f\Big(\frac{X_i+X_k}{2},\frac{X_j+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_k))=1}$$
$$\mathbf{1}_{\Phi(B(X_k\to X_i))>1}\mathbf{1}_{\Phi(B(X_l\to X_j))=1}\mathbf{1}_{\Phi(B(X_j\to X_l)=1}\mathbf{1}_{X_j^1<X_l^1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+z}{2},\frac{y+u}{2})e^{-\lambda(|B(x\to z)\cup B(u\to y)\cup B(y\to u)|)}$$
$$(1-e^{-\lambda(|B(z\to x)\setminus B(x\to z)|)})\mathbf{1}_{y_1<u_1}dxdydzdu.$$

Applying the change of variable $a=\frac{x+z}{2}$, $b=\frac{y+u}{2}$, with u and z fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 4f\Big(a,b\Big)$$
$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-z\to z)\cup B(u\to 2b-u)\cup B(2b-u\to u)|)}$$
$$(1-e^{-\lambda(|B(z\to 2a-z)\setminus B(2a-z\to z)|)})\mathbf{1}_{2b_1-u_1<u_1}dzdu\,dadb,$$

where

$$g(a,b) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-z\to z)\cup B(u\to 2b-u)\cup B(2b-u\to u)|)}$$
$$(1-e^{-\lambda(|B(z\to 2a-z)\setminus B(2a-z\to z)|)})\mathbf{1}_{2b_1-u_1<u_1}dzdu.$$

(d) $X_k \to X_i$, but X_i is left most in (X_i, X_k) and $X_l \to X_j$, but X_j is left most in (X_j, X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k,X_l\in\Phi, i\neq j\neq k\neq l} f\Big(\frac{X_i+X_k}{2}, \frac{X_j+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_k))=1}$$

$$\mathbf{1}_{\Phi(B(X_k\to X_i))=1}\mathbf{1}_{\Phi(B(X_l\to X_j))=1}\mathbf{1}_{\Phi(B(X_j\to X_l)=1}\mathbf{1}_{X_i^1<X_k^1}\mathbf{1}_{X_j^1<X_l^1}\Big]$$

$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+z}{2}, \frac{y+u}{2})e^{-\lambda(|B(x\to z)\cup B(z\to x)\cup B(u\to y)\cup B(y\to u)|)}$$

$$\mathbf{1}_{x_1<z_1}\mathbf{1}_{y_1<u_1}dxdydzdu.$$

Applying the change of variable $a = \frac{x+z}{2}$, $b = \frac{y+u}{2}$, with u and z fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 4f\Big(a, b\Big)$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-z\to z)\cup B(z\to 2a-z)\cup B(u\to 2b-u)\cup B(2b-u\to u)|)}$$

$$\mathbf{1}_{2a_1-z_1<z_1}\mathbf{1}_{2b_1-u_1<u_1}dzdu\, dadb,$$

where

$$g(a,b) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-z\to z)\cup B(z\to 2a-z)\cup B(u\to 2b-u)\cup B(2b-u\to u)|)}$$

$$\mathbf{1}_{2a_1-z_1<z_1}\mathbf{1}_{2b_1-u_1<u_1}dzdu.$$

It's clear from the expressions of densities $g(a, b)$ that they are bounded in each case.

The cases listed are exhaustive and disjoint. This proves the result.

□

Third Order Moment Measure

Lemma A.4. Φ_1 *has a third factorial moment measure with a bounded density.*

Proof. In a manner similar to the second factorial moment measure, we want to show that

$$\mathbb{E}\Big[\sum_{x_i\neq x_j\neq x_k\in\tilde{\Phi}_1} f(x_i,x_j,x_k)\Big] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x,y,z)g(x,y,z)dxdydz,$$

with $g(x,u,z)$ being the density of the 3^{rd} factorial moment measure.

By $\tilde{\Phi}_1$, we denote the point process of Φ at step 1 without the multiplicity. Let $\tilde{\Phi}$ be the Poison point process where one removes the leftmost point of each ultimate leader pair of order 0. There is a bijection between $\tilde{\Phi}$ and $\tilde{\Phi}_1$.

We analyze different possible configurations and show that there is a density of the third moment measure in each disjoint case.

1. A triple of points of $\tilde{\Phi}$ that follow yet another point, which is the same case as a triple of points of Φ that follow yet another point.

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\tilde{\Phi},i\neq j\neq k}\sum_{X_l\in\Phi} f(\frac{X_i+X_l}{2},\frac{X_j+X_l}{2},\frac{X_k+X_l}{2})\mathbf{1}_{\Phi(B(X_i\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_j\to X_l))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+u}{2},\frac{y+u}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to u)\cup B(y\to u)\cup B(z\to u)|)}dxdydzdu,$$

where we use the fact that $\tilde{\Phi}=\Phi$ and is a Poisson point process. Applying the change of variable $a=\frac{x+u}{2}$, $b=\frac{y+u}{2}$, $c=\frac{z+u}{2}$ and u fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-u\to u)\cup B(2b-u\to u)\cup B(2c-u\to u)|)}dadbdcdu,$$

where

$$g(a,b,c) = \int_{\mathbb{R}^2} e^{-\lambda(|B(2a-u\to u)\cup B(2b-u\to u)\cup B(2c-u\to u)|)} du.$$

2. $X_i \to X_j$, $X_j \to X_k$, $X_k \to X_l$ and then there are two cases

   (a) $X_l \not\to X_k$

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\tilde{\Phi},i\neq j\neq k}\sum_{X_l\in\Phi} f\Big(\frac{X_i+X_j}{2},\frac{X_j+X_k}{2},\frac{X_k+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_j))=1}$$
$$\mathbf{1}_{\Phi(B(X_j\to X_k))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}\mathbf{1}_{\Phi(X_l\to X_k))>1}\Big]$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2},\frac{y+z}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to y)\cup B(y\to z)\cup B(z\to u)|)}\Big(1-e^{-\lambda(|B(u\to z)\setminus B(z\to u)|)}\Big)dxdydzdu,$$

Applying the change of variables $a=\frac{x+y}{2}$, $b=\frac{y+z}{2}$, $c=\frac{z+u}{2}$ and u is fixed. We get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-2b+2c-u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)dadbdcdu,$$

where

$$g(a,b,c) = \int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+2c-u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)du.$$

(b) $X_l \to X_k$, but X_k is the left most point in (X_k, X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\tilde{\Phi},i\neq j\neq k}\sum_{X_l\in\Phi} f\Big(\frac{X_i+X_j}{2},\frac{X_j+X_k}{2},\frac{X_k+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_j))=1}$$

$$\mathbf{1}_{\Phi(B(X_j\to X_k))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}\mathbf{1}_{\Phi(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2},\frac{y+z}{2},\frac{z+u}{2})$$

$$e^{-\lambda(|B(x\to y)\cup B(y\to z)\cup B(z\to u)\cup B(u\to z)|)}\mathbf{1}_{z_1<u_1}dxdydzdu,$$

Applying the change of variables $a=\frac{x+y}{2}$, $b=\frac{y+z}{2}$, $c=\frac{z+u}{2}$ and u is fixed. We get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$

$$e^{-\lambda(|B(2a-2b+2c-u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$

$$\mathbf{1}_{2c_1-u_1<u_1}dadbdcdu,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+2c-u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$

$$\mathbf{1}_{2c_1-u_1<u_1}du.$$

3. $X_i \to X_k$, $X_j \to X_k$, $X_k \to X_l$ and then there are two additional cases:

(a) $X_l \not\to X_k$. Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\tilde{\Phi},i\neq j\neq k}\sum_{X_l\in\Phi} f\Big(\frac{X_i+X_k}{2},\frac{X_j+X_k}{2},\frac{X_k+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_k))=1}$$

$$\mathbf{1}_{\Phi(B(X_j\to X_k))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}\mathbf{1}_{\Phi(X_l\to X_k))>1}\Big]$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+z}{2},\frac{y+z}{2},\frac{z+u}{2})$$

$$e^{-\lambda(|B(x\to z)\cup B(y\to z)\cup B(z\to u)|)}\Big(1-e^{-\lambda(|B(u\to z)\setminus B(z\to u)|)}\Big)dxdydzdu,$$

Applying the change of variables $a = \frac{x+z}{2}$, $b = \frac{y+z}{2}$, $c = \frac{z+u}{2}$ and u is fixed. We get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-2c+u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)dadbdcdu,$$

where

$$g(a,b,c) = \int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2c+u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)du.$$

(b) $X_l \to X_k$, but X_k is the left most point in (X_k, X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\tilde{\Phi},i\neq j\neq k}\sum_{X_l\in\Phi} f\Big(\frac{X_i+X_k}{2},\frac{X_j+X_k}{2},\frac{X_k+X_l}{2}\Big)\mathbf{1}_{\Phi(B(X_i\to X_k))=1}$$
$$\mathbf{1}_{\Phi(B(X_j\to X_k))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}\mathbf{1}_{\Phi(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+z}{2},\frac{y+z}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to z)\cup B(y\to z)\cup B(z\to u)\cup B(u\to z)|)}\mathbf{1}_{z_1<u_1}dxdydzdu,$$

Applying the change of variables $a = \frac{x+z}{2}$, $b = \frac{y+z}{2}$, $c = \frac{z+u}{2}$ and u is fixed. We get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-2c+u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2c_1-u_1<u_1}dadbdcdu,$$

where

$$g(a,b,c) = \int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2c+u\to 2b-2c+u)\cup B(2b-2c+u\to 2c-u)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2c_1-u_1<u_1} du.$$

4. $X_i \to X_j$, $X_j \to X_s$, $X_k \to X_l$. There are four cases we need to analyze.

   (a) $X_s \not\to X_j$ and $X_l \not\to X_k$. Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\tilde{\Phi},i\neq j\neq k}\sum_{X_l,X_s\in\Phi,l\neq s} f(\frac{X_i+X_j}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_j))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_s\to X_j))>1}\mathbf{1}_{\Phi(B(X_l\to X_k))>1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to y)\cup B(y\to s)\cup B(z\to u)|)}\Big(1-e^{-\lambda(|B(u\to z)\setminus B(z\to u)|)}\Big)$$
$$\Big(1-e^{-\lambda(|B(s\to y)\setminus B(y\to s)|)}\Big)dxdydzduds,$$

   Applying the change of variable $a=\frac{x+y}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(2c-u\to u)|)}\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dadbdcduds,$$

   where

$$g(a,b,c) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)duds.$$

(b) $X_s \to X_j$, but X_j is left most of (X_j, X_s), and $X_l \not\to X_k$. Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\tilde{\Phi},i\neq j\neq k}\sum_{X_l,X_s\in\Phi,l\neq s} f(\frac{X_i+X_j}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_j))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_s\to X_j))=1}\mathbf{1}_{X_j^1<X_s^1}\mathbf{1}_{\Phi(B(X_l\to X_k))>1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to y)\cup B(y\to s)\cup B(s\to y)\cup B(z\to u)|)}\mathbf{1}_{y_1<s_1}$$
$$\Big(1-e^{-\lambda(|B(u\to z)\setminus B(z\to u)|)}\Big)dxdydzduds,$$

Applying the change of variable $a=\frac{x+y}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)|)}\mathbf{1}_{2b_1-s_1<s_1}$$
$$\Big(1-e^{-\lambda(|B(u\to 2z-u)\setminus B(2z-u\to u)|)}\Big)dadbdcduds,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2b_1-s_1<s_1}\Big(1-e^{-\lambda(|B(u\to 2z-u)\setminus B(2z-u\to u)|)}\Big)duds.$$

(c) $X_s \not\to X_j$ and $X_l \to X_k$, but X_k is left most of (X_k, X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi,i\neq j\neq k}\sum_{X_l,X_s\in\Phi,l\neq s} f(\frac{X_i+X_j}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_j))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_s\to X_j))>1}\mathbf{1}_{\Phi(B(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to y)\cup B(y\to s)\cup B(z\to u)\cup B(u\to z)|)}\mathbf{1}_{z_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(s\to y)\setminus B(y\to s)|)}\Big)dxdydzduds,$$

Applying the change of variable $a=\frac{x+y}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}\mathbf{1}_{2c_1-u_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dadbdcduds,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2c_1-u_1<u_1}\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)duds.$$

(d) $X_s \to X_j$, but X_j is left most of (X_j, X_s) and $X_l \to X_k$, but X_k is left most of (X_k, X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi, i\neq j\neq k}\ \sum_{X_l,X_s\in\Phi, l\neq s} f(\frac{X_i+X_j}{2}, \frac{X_j+X_s}{2}, \frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_j))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_s\to X_j))=1}\mathbf{1}_{X_j^1<X_s^1}\mathbf{1}_{\Phi(B(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+y}{2}, \frac{y+s}{2}, \frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to y)\cup B(y\to s)\cup B(s\to y)\cup B(z\to u)\cup B(u\to z)|)}$$
$$\mathbf{1}_{y_1<u_1}\mathbf{1}_{z_1<u_1}\Big) dxdydzduds,$$

Applying the change of variable $a = \frac{x+y}{2}$, $b = \frac{y+s}{2}$, $c = \frac{z+u}{2}$, with u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a, b, c\Big)$$
$$e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2b_1-s_1<s_1}\mathbf{1}_{2c_1-u_1<u_1}\Big) da\, db\, dc\, du\, ds,$$

where

$$g(a,b,c) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-2b+s\to 2b-s)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2b_1-s_1<s_1}\mathbf{1}_{2c_1-u_1<u_1}\Big) duds.$$

5. $X_i \to X_t$, $X_j \to X_s$, $X_k \to X_l$, then we have the following disjoint cases

(a) $X_t \not\to X_i$, $X_s \not\to X_j$ and $X_l \not\to X_k$. Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi, i\neq j\neq k}\ \sum_{X_l,X_s,X_t\in\Phi, l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_t\to X_i))>1}\mathbf{1}_{\Phi(B(X_s\to X_j))>1}\mathbf{1}_{\Phi(B(X_l\to X_k))>1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to t)\cup B(y\to s)\cup B(z\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to x)\setminus B(x\to t)|)}\Big)\Big(1-e^{-\lambda(|B(u\to z)\setminus B(z\to u)|)}\Big)$$
$$\Big(1-e^{-\lambda(|B(s\to y)\setminus B(y\to s)|)}\Big)dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to 2a-t)\setminus B(2a-t\to t)|)}\Big)\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to 2a-t)\setminus B(2a-t\to t)|)}\Big)\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dudsdt.$$

(b) $X_t \to X_i$, but X_i is the left most of (X_i, X_t), $X_s \not\to X_j$ and $X_l \not\to X_k$.

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi,i\neq j\neq k}\ \sum_{X_l,X_s,X_t\in\Phi,l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_t\to X_i))=1}\mathbf{1}_{X_i^1<X_t^1}\mathbf{1}_{\Phi(B(X_s\to X_j))>1}\mathbf{1}_{\Phi(B(X_l\to X_k))>1}\Big]$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to t)\cup B(t\to x)\cup B(y\to s)\cup B(z\to u)|)}$$
$$\mathbf{1}_{x_1<t_1}\Big(1-e^{-\lambda(|B(u\to z)\setminus B(z\to u)|)}\Big)$$
$$\Big(1-e^{-\lambda(|B(s\to y)\setminus B(y\to s)|)}\Big)dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)\mathbf{1}_{2a_1-t_1<t_1}$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)\mathbf{1}_{2a_1-t_1<t_1}$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dudsdt.$$

(c) $X_t\not\to X_i$, $X_s\to X_j$, but X_j is left most of (X_j,X_s), and $X_l\not\to X_k$.

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi,i\neq j\neq k}\ \sum_{X_l,X_s,X_t\in\Phi,l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_t\to X_i))>1}\mathbf{1}_{\Phi(B(X_s\to X_j))=1}\mathbf{1}_{X_j^1<X_s^1}\mathbf{1}_{\Phi(B(X_l\to X_k))>1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to t)\cup B(y\to s)\cup B(s\to y)\cup B(z\to u)|)}\mathbf{1}_{y_1<s_1}$$
$$\Big(1-e^{-\lambda(|B(t\to x)\backslash B(x\to t)|)}\Big)\Big(1-e^{-\lambda(|B(u\to z)\backslash B(z\to u)|)}\Big)dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)|)}\mathbf{1}_{2c_1-u_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(t\to 2a-t)\backslash B(2a-t\to t)|)}\Big)$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\backslash B(2c-u\to u)|)}\Big)dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to 2a-t)\backslash B(2a-t\to t)|)}\Big)\mathbf{1}_{2c_1-u_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\backslash B(2c-u\to u)|)}\Big)dudsdt.$$

(d) $X_t\not\to X_i$, $X_s\not\to X_j$, and $X_l\to X_k$, but X_k is left most of (X_k,X_l).

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi,i\neq j\neq k}\sum_{X_l,X_s,X_t\in\Phi,l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_t\to X_i))>1}\mathbf{1}_{\Phi(B(X_s\to X_j))>1}\mathbf{1}_{\Phi(B(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to t)\cup B(y\to s)\cup B(z\to u)\cup B(u\to z)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to x)\setminus B(x\to t)|)}\Big)\mathbf{1}_{z_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(s\to y)\setminus B(y\to s)|)}\Big)dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to 2a-t)\setminus B(2a-t\to t)|)}\Big)\mathbf{1}_{2c_1-u_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to 2a-t)\setminus B(2a-t\to t)|)}\Big)\mathbf{1}_{2c_1-u_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dudsdt.$$

(e) $X_t\not\to X_i$, $X_s\to X_j$, but X_j is left most of (X_j,X_s), and $X_l\to X_k$, but X_k is left most of (X_k, X_l).

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi,i\neq j\neq k}\ \sum_{X_l,X_s,X_t\in\Phi,l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_t\to X_i))>1}\mathbf{1}_{\Phi(B(X_s\to X_j))=1}\mathbf{1}_{X_j^1<X_s^1}\mathbf{1}_{\Phi(B(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to t)\cup B(y\to s)\cup B(s\to y)\cup B(z\to u)\cup B(u\to z)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to x)\setminus B(x\to t)|)}\Big)\mathbf{1}_{y_1<s_1}\mathbf{1}_{z_1<u_1}dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\Big(1-e^{-\lambda(|B(t\to 2a-t)\setminus B(2a-t\to t)|)}\Big)$$
$$\mathbf{1}_{2c_1-u_1<u_1}\mathbf{1}_{2b_1-s_1<s_1}dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(1-e^{-\lambda(|B(t\to 2a-t)\setminus B(2a-t\to t)|)}\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2c_1-u_1<u_1}\mathbf{1}_{2b_1-s_1<s_1}dudsdt.$$

(f) $X_t\to X_i$, but X_i is the left most of (X_i, X_t), $X_s \not\to X_j$ and $X_l \to X_k$, but X_k is left most of (X_k, X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi,i\neq j\neq k}\ \sum_{X_l,X_s,X_t\in\Phi,l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_t\to X_i))=1}\mathbf{1}_{X_i^1<X_t^1}\mathbf{1}_{\Phi(B(X_s\to X_j))>1}\mathbf{1}_{\Phi(B(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to t)\cup B(t\to x)\cup B(y\to s)\cup B(z\to u)\cup B(u\to z)|)}$$
$$\mathbf{1}_{x_1<t_1}\mathbf{1}_{z_1<u_1}\Big(1-e^{-\lambda(|B(s\to y)\setminus B(y\to s)|)}\Big)dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\mathbf{1}_{2a_1-t_1<t_1}\mathbf{1}_{2c_1-u_1<u_1}$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbf{1}_{2a_1-t_1<t_1}\mathbf{1}_{2c_1-u_1<u_1}$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$
$$\Big(1-e^{-\lambda(|B(s\to 2b-s)\setminus B(2b-s\to s)|)}\Big)dudsdt.$$

(g) $X_t\to X_i$, but X_i is the left most of (X_i,X_t), $X_s\to X_j$, but X_j is left most of (X_j,X_s) and $X_l\not\to X_k$. Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi, i\neq j\neq k}\ \sum_{X_l,X_s,X_t\in\Phi, l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$
$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$
$$\mathbf{1}_{\Phi(B(X_t\to X_i))=1}\mathbf{1}_{X_i^1<X_t^1}\mathbf{1}_{\Phi(B(X_s\to X_j))=1}\mathbf{1}_{X_j^1<X_s^1}\mathbf{1}_{\Phi(B(X_l\to X_k))>1}\Big]$$
$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$
$$e^{-\lambda(|B(x\to t)\cup B(t\to x)\cup B(y\to s)\cup B(z\to u)|)}$$
$$\mathbf{1}_{x_1<t_1}\mathbf{1}_{y_1<s_1}$$
$$\Big(1-e^{-\lambda(|B(u\to z)\setminus B(z\to u)|)}\Big)dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$
$$e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)|)}$$
$$\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)\mathbf{1}_{2a_1-t_1<t_1}\mathbf{1}_{2b_1-s_1<s_1}dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Big(1-e^{-\lambda(|B(u\to 2c-u)\setminus B(2c-u\to u)|)}\Big)\mathbf{1}_{2a_1-t_1<t_1}$$
$$\mathbf{1}_{2b_1-s_1<s_1}e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)|)}dudsdt.$$

(h) $X_t\to X_i$, but X_i is the left most of (X_i,X_t), $X_s\to X_j$, but X_j is left most of (X_j,X_s) and $X_l\to X_k$, but X_k is left most of (X_k,X_l). Then

$$\mathbb{E}\Big[\sum_{X_i,X_j,X_k\in\Phi,i\neq j\neq k}\ \sum_{X_l,X_s,X_t\in\Phi,l\neq s\neq t} f(\frac{X_i+X_t}{2},\frac{X_j+X_s}{2},\frac{X_k+X_l}{2})$$

$$\mathbf{1}_{\Phi(B(X_i\to X_t))=1}\mathbf{1}_{\Phi(B(X_j\to X_s))=1}\mathbf{1}_{\Phi(B(X_k\to X_l))=1}$$

$$\mathbf{1}_{\Phi(B(X_t\to X_i))=1}\mathbf{1}_{X_i^1<X_t^1}\mathbf{1}_{\Phi(B(X_s\to X_j))=1}\mathbf{1}_{X_j^1<X_s^1}\mathbf{1}_{\Phi(B(X_l\to X_k))=1}\mathbf{1}_{X_k^1<X_l^1}\Big]$$

$$=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(\frac{x+t}{2},\frac{y+s}{2},\frac{z+u}{2})$$

$$e^{-\lambda(|B(x\to t)\cup B(t\to x)\cup B(y\to s)\cup B(s\to y)\cup B(z\to u)\cup B(u\to z)|)}$$

$$\mathbf{1}_{x_1<t_1}\mathbf{1}_{y_1<s_1}\mathbf{1}_{z_1<u_1}dxdydzdudsdt,$$

Applying the change of variable $a=\frac{x+t}{2}$, $b=\frac{y+s}{2}$, $c=\frac{z+u}{2}$, with t, u and s fixed, we get

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} 8f\Big(a,b,c\Big)$$

$$e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}$$

$$\mathbf{1}_{2a_1-t_1<t_1}\mathbf{1}_{2b_1-s_1<s_1}\mathbf{1}_{2c_1-u_1<u_1}dadbdcdudsdt,$$

where

$$g(a,b,c)=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbf{1}_{2a_1-t_1<t_1}\mathbf{1}_{2b_1-s_1<s_1}\mathbf{1}_{2c_1-u_1<u_1}$$

$$e^{-\lambda(|B(2a-t\to t)\cup B(t\to 2a-t)\cup B(2b-s\to s)\cup B(s\to 2b-s)\cup B(2c-u\to u)\cup B(u\to 2c-u)|)}dudsdt.$$

It's clear from the expressions of densities $g(a,b,c)$ that they are bounded in each case. Like for the second order moment measure, this list is exhaustive and disjoint. This proves the result.

□

Appendix B

Chapter 3 Appendix

In this section we give all the proofs that were not included in Chapter 3. First we show in Section 3.B.1 that there are no other ways of forming an ultimate leader pairs of order 1.

B.1 Existence of only 2 types of ultimate leader pairs of order 1

We want to show that a situation like in Figure B.1, where two agents that follow different leaders become an ultimate leader pair of order 1, is impossible. In order to prove that, we look into the general geometric properties of the follower/leader pair after one step.

Before we continue with the analysis, we need to introduce the notion of *convex hull*. The convex hull of a set in $\mathbb{R}^2$ is the smallest convex set that contains it. For example, the convex hull of the two disks which is shown in Figure B.2 is the union of the two disks (one centered at A with radius $|AB|$ and other centered at B with radius $|BC|$), and the shaded region in the Figure. *The enlargement region* is the set difference between the convex hull and the union of the disks. The enlargement region is the shaded region in Figure B.2. When we refer to the convex hull of a follower/leader pair, we mean the convex hull of the two disks centered at the follower/leader, and with radii that are the distances to their respective leaders.

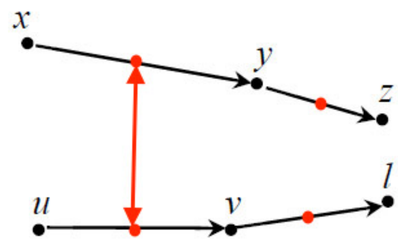


Figure B.1: The initial agent configuration is in black. Step 1 is in red. Example of an ultimate leader pair of order 1 formed from agents that have different leaders.

In the lemmas that follow, we give a necessary condition for a leader swap.

In the example in Figure B.1, agents z and l are depicted as different agents but the analysis below does not exclude situations where $z = l$, or WLOG $v = z = l$.

Lemma B.1. *Take a follower/leader pair. A necessary condition for the follower to experience a leader swap in one step is the presence of another agent at step 1 in the convex hull of the follower/leader pair.*

Proof. The proof of this lemma relies on simple geometry and looking at a special case and then showing that the same argument holds for the general setting. By special case we mean the case when the agents are collinear. Pick a triple of agents, A, B and C such that, A follows B and B follows C. We draw a circle around A with radius $|AB| = R_1$ and a circle around B with radius $|BC| = R_2$ as in Figure B.2. The positions of A and B at time step 1 are A' and B' respectively, with $A' = \frac{A+B}{2}$ and $B' = \frac{B+C}{2}$. We now show that the ball $B(A', |A'B'|)$ is inside the convex hull. In the special case, where points A, B and C are colinear, $|A'B'| = \frac{1}{2}(R_1 + R_2)$. Looking at Figure B.2,

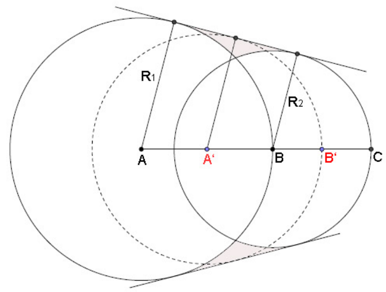


Figure B.2: Special case where the agent positions are collinear. Point A is the follower of B. Point B is the follower of C. Next time step we denote the positions as A' and B'. Clearly disk around A' with radius $|A'B'|$ is inside the convex hull of two disks, one centered at A and radius $|AB|$ and other centered at B with radius $|BC|$. Shaded region is what we refer to as the enlargement area.

we can see that by classical properties of trapezoids, the distance from A' to the boundary of the convex hull is exactly $\frac{1}{2}(R_1 + R_2)$. So, in order for A to swap leader, there needs to be another agent in the ball centered at A' and with radius $|A'B'|$ at step 1. This requires that another agent needs to be in the convex hull of the A/B pair at step 1. The general case where A, B and C are not collinear is depicted in Figure B.3. Then by the triangle inequality, the distance $|A'B'| < \frac{1}{2}(R_1 + R_2)$. Therefore, by the same argument as above, A can still only swap leaders if there is an agent at step 1 inside the convex hull of A/B. It is clearly a necessary but not a sufficient condition.

□

B.2 Algorithm of Integral Geometric Frequency

Algorithm of the upper bound calculation of frequency of the ultimate leader pairs of order 1, type 1 can be found below.

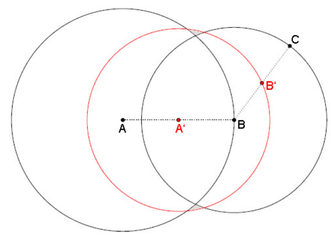


Figure B.3: General case. Point A is the follower of B. Point B is the follower of C. Three agents are not necessarily in one line. For the next step we denote the positions of A as A' and B as B'. Clearly disk around A' with radius $|A'B'|$ is inside the convex hull of two disks, one centered at A and radius $|AB|$ and other centered at B with radius $|BC|$.

Python Code:

```
import numpy as np
import scipy as sp

a = 10
b = 0.2

# create a grid with separation given by b
# this gives [0, b, 2b, 3b, .... , a]
x = np.arange(0, a+b, b)

# keep x3, y3 fixed at the center
x3, y3 = 0.5*a, 0.5*a

# create an array for x4, y4
x4, y4 = np.meshgrid(x,x)
x4, y4 = x4.ravel(), y4.ravel()
#halfway points for x3,y3,x4,y4
x34, y34 = (x3 + x4)/2 , (y3 + y4)/2

# create vector of distances between x3,y3 and x4, y4
d34 = np.linalg.norm( np.c_[x3, y3] - np.c_[x4, y4], axis=1)
```

```
n_random = 1000
exp_area = 0.

# loop over x1,y1,x2,y2
for x1 in x:
  for y1 in x:
     for x2 in x:
        for y2 in x:
            d13 = np.linalg.norm( np.c_[x1, y1] - np.c_[x3, y3])
            d12 = np.linalg.norm( np.c_[x1, y1] - np.c_[x2, y2])
            d23 = np.linalg.norm( np.c_[x2, y2] - np.c_[x3, y3])
# if 1's distance to 2 is smaller than its distance to 3, skip
# if 2's distance to 1 is smaller than its distance to 3, skip
# if 1's distance to 3 is leq 2's distance to 3, skip. why??
            if d13 >= d12 or d23 >= d12 or (d13 <= d23):
                 continue

    # halfway points
            x13, y13 = (x1+x3)/2, (y1 + y3)/2
            x23, y23 = (x2 + x3)/2, (y2 + y3)/2

 d12_2 = np.linalg.norm( np.c_[x13, y13] - np.c_[x23, y23],axis=1)
 d13_2 = np.linalg.norm(np.c_[x13, y13] - np.c_[x34, y34],axis=1)
 d23_2 = np.linalg.norm(np.c_[x23, y23] - np.c_[x34, y34],axis=1)


    # now compute distance to x4, y4
 d14 = np.linalg.norm( np.c_[x1, y1] - np.c_[x4, y4], axis=1)
 d24 = np.linalg.norm( np.c_[x2, y2] - np.c_[x4, y4], axis=1)

    # find conditions for the points being nearest neigbors
    # & denotes the AND operation
           xx1 = (d13< d12) & (d13 < d14) # 1's nn is 3
           xx2 = (d23 < d12) & (d23 < d24) # 2's nn is 3
           xx3 = (d34 < d13) & (d34 < d23) # 4's nn is 3
           # condition for halfway points
           xx4 = (d12_2 < d13_2) & (d12_2 < d23_2)
```

```
    # find numeric indices of points that satisfy ALL conditions
            xx = xx1 & xx2 & xx3 & xx4
            xx_idx = np.where(xx == True)[0]

            for idx in xx_idx:
                random = a*np.random.rand(n_random, 2)
 d_random_1 = np.linalg.norm(random - np.c_[x1,y1], axis=1)
 d_random_2 = np.linalg.norm(random - np.c_[x2,y2], axis=1)
 d_random_3 = np.linalg.norm(random - np.c_[x3,y3], axis=1)
    # check if it lies in any of the circles. | denotes OR
xx_center = (d_random_1 < d13 ) | (d_random_2 < d23)
 | (d_random_3 < d34[idx])

  temp_area = np.sum(xx_center)*(a**2)/n_random
  exp_area += np.exp(-1*temp_area)


%print('Overcounted area = %f'%(exp_area))
```

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